

THE CONSTANT IN THOMAE-TYPE FORMULAS FOR EIGHT POINTS ON THE COMPLEX PROJECTIVE LINE

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ABSTRACT. We consider the family of cyclic fourfold covers $C(x)$ of the complex projective line branched at eight points $x = (x_1, \dots, x_8)$. The period map identifies the configuration space $X(2, 8)$ of the branch points with a Zariski open subset of a quotient of the five-dimensional complex ball. It is known that the inverse of the period map is expressed projectively by 105 automorphic forms f_J , which are proportional to the signed branch-point polynomials $\hat{x}_J$ with a common scalar factor. We normalize an element ω of the one-dimensional eigenspace of the deck transformation group acting on the holomorphic differentials on $C(x)$. For the period η of ω , we determine the common scalar factor: it is the product of a constant expressed in terms of the gamma function and the square of a quadratic form in η .

1. INTRODUCTION

For a hyperelliptic curve $y^2 = \prod_j (z - x_j)$, Thomae's classical formula relates powers of the nonzero theta constants at its period matrix to products of differences of the branch points x_j and a determinant of periods; see [Fay73, Proposition 3.6]. A generalization to the Z_N curves $s^N = \prod_j (z - \lambda_j)$, where the λ_j are distinct and the degree of the polynomial is divisible by N , is proved in [Nak97]. The period maps of several families of cyclic covers of $\mathbb{P}^1$ take values in complex balls in the sense of Deligne–Mostow [DM86] and Terada [Ter83]. Thomae-type formulas are established for such families; see [MS10, Theorem 2.1] and [Shi11, Theorem 2.2]. In particular, a Thomae-type formula is established in [MN26] for the cyclic fourfold covers of $\mathbb{P}^1$ attached to the Deligne–Mostow admissible sequence $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4})$, where the period map takes values in a three-dimensional complex ball.

We study the family of cyclic fourfold covers

$$C(x): \quad w^4 = \prod_{j=1}^7 (z - x_j)$$

of $\mathbb{P}^1$ branched at $x_1, \dots, x_7, x_8 = \infty$. The configuration space $X(2, 8)$ of the branch points is defined as the quotient

$$X(2, 8) = \mathrm{PGL}(2, \mathbb{C}) \backslash \{(x_1, \dots, x_8) \in (\mathbb{P}^1)^8 \mid x_j \neq x_k \ (j \neq k)\},$$

where $\mathrm{PGL}(2, \mathbb{C})$ acts diagonally on $(\mathbb{P}^1)^8$. We represent a point of $X(2, 8)$ by $x = (x_1, \dots, x_8)$. Sending (x_1, x_7, x_8) to $(0, 1, \infty)$ by an element of $\mathrm{PGL}(2, \mathbb{C})$, we obtain a complete set

$$\{(0, x_2, \dots, x_6, 1, \infty) \in (\mathbb{P}^1)^8 \mid x_j \neq 0, 1 \ (2 \leq j \leq 6), \ x_j \neq x_k \ (2 \leq j < k \leq 6)\}$$

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of representatives for $X(2, 8)$. We write $\varpi: \tilde{X}(2, 8) \rightarrow X(2, 8)$ for the universal cover of $X(2, 8)$. By [MY93, Section 4] and [MT04, Proposition 3.5], the period map *per* identifies $X(2, 8)$ with a Zariski open subset of the ball quotient $\Gamma_M \backslash \mathbb{B}_5$. The period map has a holomorphic lift $\widetilde{\text{per}}: \tilde{X}(2, 8) \rightarrow \mathbb{B}_5$. Here $\mathbb{B}_5$ is the five-dimensional complex ball defined by the quotient

$$\mathcal{B}/\mathbb{C}^* = \{\eta \in \mathbb{C}^6 \mid {}^t\bar{\eta}U\eta < 0\}/\mathbb{C}^* = \{\eta \in \mathbb{P}^5 \mid {}^t\bar{\eta}U\eta < 0\},$$

where

$$U = \begin{pmatrix} I_2 & & \\ & -1 & \\ & -1 & \\ & & I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and Γ_M is the conjugate $T\Gamma(1+i)T^{-1}$ of the principal congruence subgroup $\Gamma(1+i)$ of level $1+i$ of the Picard modular group Γ over $\mathbb{Z}[i]$ by T ; see Section 2 for the definitions of $\Gamma(1+i)$ and $T \in \text{GL}(6, \mathbb{Q}[i])$. We write Γ for Euler's gamma function and Γ for the Picard modular group. Note that the period $\eta = \text{per}(x)$ can be lifted to a value in $\mathcal{B}$ by taking a path in $X(2, 8)$, and we write $\eta(\tilde{x})$ for the value at $\tilde{x} \in \tilde{X}(2, 8)$. Matsumoto and Terasoma [MT04] constructed 105 automorphic forms $T_J^{(2)}$ on $\mathbb{B}_5$, indexed by the $(2, 2, 2, 2)$ -partitions $J = \langle j_1j_2; j_3j_4; j_5j_6; j_7j_8 \rangle$ of $\{1, \dots, 8\}$, that is, by the partitions of $\{1, \dots, 8\} = \{j_1, \dots, j_8\}$ into four blocks $\{j_1, j_2\}, \{j_3, j_4\}, \{j_5, j_6\}, \{j_7, j_8\}$ with $j_{2k-1} < j_{2k}$. They proved in [MT04, Theorem 6.4] that the projective point defined by these forms coincides, through the period map, with the projective point defined by the 105 polynomials

$$x_J = \prod_{k=1}^4 (x_{j_{2k-1}} - x_{j_{2k}}),$$

where $x = (x_1, \dots, x_8) \in X(2, 8)$ and the factor containing ∞ is omitted. Matsumoto, Minowa, and Nishimura [MMN07] expressed all 105 forms, denoted by f_J , as explicit polynomials in theta constants. In their tables, to each form f_J there corresponds the polynomial $\hat{x}_J = \varepsilon_J x_J$ with $\varepsilon_J \in \{\pm 1\}$. The results of [MT04] and [MMN07] determine the vector $(f_J(\eta))_J$ up to a common scalar factor: this vector is proportional to $(\hat{x}_J(x))_J$. In this paper, we determine this common scalar factor. Our main result is the following.

Main Theorem. *Let $\tilde{x} \in \tilde{X}(2, 8)$, and put $x = \varpi(\tilde{x})$ and $\eta = \eta(\tilde{x}) \in \mathcal{B}$. Then, for every $(2, 2, 2, 2)$ -partition J , we have*

$$(1.1) \quad f_J(\eta) = \kappa ({}^t\eta U \eta)^2 \hat{x}_J(x), \quad \kappa = -\frac{\vartheta_{00}(i)^{16}}{(8\pi)^4} = -\frac{1}{2^{12} \Gamma(3/4)^{16}}.$$

Here we use the formula $\vartheta_{00}(i) = \pi^{1/4}/\Gamma(3/4)$ to express κ in terms of $\Gamma(3/4)$. The constant κ permits comparison with other Thomae-type formulas and with boundary values; see Remark 5.2 for the six-point case of [MN26]. We also decompose a theta series on $\mathfrak{S}_4$ arising from the limit curve of Section 4 into theta constants on the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$; see Proposition 4.3.

We explain our method. The proof has two parts. First, we show that for each J the ratio of $f_J(\eta)$ to $({}^t\eta U \eta)^2 \hat{x}_J(x)$ is constant on $X(2, 8)$, and that this constant does not depend on J ; see Proposition 3.4. The invariance of this ratio under the monodromy follows from the automorphy of the forms f_J , which is established in [MMN07]. The constancy follows from Hartogs' theorem, since every holomorphic

function on $\mathbb{P}^5$ is constant. At a boundary hyperplane where two points coincide, Deligne–Mostow’s extension results give, on a finite local cover, a holomorphic extension of the period η . Since the ratio is invariant under the monodromy, the ratio descends from this cover and extends across the hyperplane. By the equality of the two projective points established in [MT04] and [MMN07], the constant does not depend on J . Second, we evaluate the constant at the degenerate configuration $(0, 0, 0, c, c, c, 1, \infty)$ with $c \in (0, 1)$. We choose this configuration so that the polynomial attached to $J^* = \langle 14; 25; 36; 78 \rangle$ does not vanish and the computation of the period reduces to an evaluation of the elliptic modular function. There the normalization of the limit curve $w^4 = z^3(z - c)^3(z - 1)$ has genus 3. The theta constants split into products of theta constants on $\mathbb{H}$ at the moduli

$$\frac{\tau^*}{2}, \quad -\frac{1}{2\tau^*}, \quad \frac{i}{2},$$

where $\tau^* = -i(u + v)/(u - v)$, and u and v are explicit period integrals of the limit curve; see Section 4 for their definitions. Using classical quadratic transformations of the Gauss hypergeometric series, we identify τ^* with a value at $m = (1 - \sqrt{c})/2$ of a Schwarz map of the Gauss hypergeometric differential equation with parameters $\frac{1}{2}, \frac{1}{2}; 1$; see Lemma 5.1. By using duplication formulas for Jacobi’s theta constants, we complete the evaluation of the constant κ ; see Sections 4 and 5.

2. PRELIMINARIES

In this section, we choose a complete set of representatives for $X(2, 8)$, recall the signs ε_J attached to the $(2, 2, 2, 2)$ -partitions, fix the branch of w as in [MT04, Section 2], and introduce the Hermitian form on $\mathbb{C}^6$. We then fix the notation for the periods and for the theta functions, define the Picard modular group and the period map, and record the automorphy of the forms f_J of [MMN07] that we use in Section 3.

2.1. The Configuration Space

The action of $\mathrm{PGL}(2, \mathbb{C})$ on $\mathbb{P}^1$ is sharply three-transitive: for any ordered triple of distinct points, there is a unique element of $\mathrm{PGL}(2, \mathbb{C})$ that carries it to $(0, 1, \infty)$. Hence a complete set of representatives for $X(2, 8)$ is given by

$$x = (x_1, \dots, x_8) = (0, x_2, x_3, x_4, x_5, x_6, 1, \infty),$$

$$(x_2, \dots, x_6) \in \{(x_2, \dots, x_6) \in \mathbb{C}^5 \mid x_j \neq 0, 1 \ (2 \leq j \leq 6), \ x_j \neq x_k \ (2 \leq j < k \leq 6)\}.$$

The coordinates $(x_2, \dots, x_6)$ identify $X(2, 8)$ with a Zariski open subset of $\mathbb{C}^5 \subset \mathbb{P}^5$. For such a representative, we consider the cover $C(x): w^4 = \prod_{j=1}^7 (z - x_j)$, whose deck transformation group is generated by

$$\rho: C(x) \ni (z, w) \mapsto (z, iw) \in C(x).$$

The form $\frac{dz}{w}$ spans the eigenspace of ρ^* for the eigenvalue $-i$ in the space of holomorphic differentials on $C(x)$, and this eigenspace is one-dimensional; cf. [MT04, Proposition 2.1]. The symmetric group S_8 acts on $x = (x_1, \dots, x_8)$ by $(\sigma \cdot x)_l = x_{\sigma^{-1}(l)}$; cf. [MMN07, Section 2.1]. For a $(2, 2, 2, 2)$ -partition J , we write $\sigma^{-1}(J)$ for the partition obtained by applying σ^{-1} to every entry of J , so that $x_J(\sigma \cdot x) = \pm x_{\sigma^{-1}(J)}(x)$. The tables of [MMN07, Section 5.1 and Theorems 1–4] attach a sign $\varepsilon_J \in \{\pm 1\}$ to

each partition. The tables list the forms f_J together with the polynomials $\pm x_J$, for instance $-x_{\langle 17;25;68;34 \rangle}$ in entry #30 of [MMN07, Theorem 2]. We put

$$\hat{x}_J = \varepsilon_J x_J.$$

We set

$$X_{\mathbb{R}} = \{(x_2, \dots, x_6) \in \mathbb{R}^5 \mid 0 < x_2 < \dots < x_6 < 1\} \subset X(2, 8),$$

and we call $X_{\mathbb{R}}$ the real chamber. We first work on $X_{\mathbb{R}}$. On the interval $\ell_j = [x_j, x_{j+1}]$ ($1 \leq j \leq 6$), we use the upper-half-plane continuation convention of [MT04, Section 2] and fix the branch of w for our normalization $x_8 = \infty$ by

$$w|_{\ell_j} = \exp\left(\frac{(7-j)\pi i}{4}\right) \left(\prod_{k \leq j} (z - x_k) \prod_{k > j} (x_k - z) \right)^{1/4},$$

where the positive real fourth root of the positive product is taken. For the branch convention with eight finite branch points, see [MT02, Section 2] and Remark 2.4. We write $\mathcal{I}_j$ for the integral $\int_{\ell_j} \frac{dz}{w}$. This branch takes the positive real fourth root on $z > 1$ and is continued through $\mathbb{H}$. If z moves from the right to the left of a branch point x_k through $\mathbb{H}$, then $\arg(z - x_k)$ changes from 0 to π . On ℓ_j , the $7 - j$ factors with $k > j$ are continued in this way, and they contribute the factor $\exp\left(\frac{(7-j)\pi i}{4}\right)$.

2.2. The Hermitian Form

We define, for $\xi, \eta \in \mathbb{C}^6$,

$$h_U(\xi, \eta) = {}^t \bar{\xi} U \eta, \quad q_U(\eta) = {}^t \eta U \eta, \quad U = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{pmatrix}.$$

Since U is real symmetric of signature $(5, 1)$, h_U is a Hermitian form of signature $(5, 1)$. The function q_U is the quadratic form associated with the symmetric bilinear form ${}^t \xi U \eta$. The function q_U appears below in the embedding of $\mathbb{B}_5$ into the Siegel upper half-space, and in (1.1) in the form ${}^t \eta U \eta$. Since we divide by q_U below, we record the following lemma.

Lemma 2.1. *If $\eta \in \mathbb{C}^6$ satisfies $h_U(\eta, \eta) < 0$, then $q_U(\eta) = {}^t \eta U \eta \neq 0$.*

PROOF. We write $\eta = \alpha + i\beta$ with real α, β . Since U is real symmetric, $h_U(\eta, \eta) = {}^t \alpha U \alpha + {}^t \beta U \beta < 0$. Suppose that $q_U(\eta) = 0$. Comparing the real and imaginary parts of $q_U(\eta)$, we obtain ${}^t \alpha U \alpha = {}^t \beta U \beta < 0$ and ${}^t \alpha U \beta = 0$. These two conditions force α and β to be linearly independent, since a linear relation between them contradicts ${}^t \alpha U \beta = 0$ together with ${}^t \alpha U \alpha < 0$. Thus α, β span a two-dimensional subspace on which U is negative definite; this contradicts the signature $(5, 1)$ of U . Hence we have $q_U(\eta) \neq 0$. $\square$

2.3. Periods and Monodromy

We recall from [MY93], [MT04], and [MMN07] the cycles, the period, and the monodromy of the family.

Fact 2.2. (1) ([MT04, Section 2.1, Proposition 2.2]) Let $x \in X_{\mathbb{R}}$. We regard each interval ℓ_j ($1 \leq j \leq 6$) as a path on $C(x)$ via the branch fixed above. The chains $A_j = (1 - \rho^2)\ell_j$ and $B_j = \rho A_j$ form a basis of the eigenspace $H_1(C(x), \mathbb{Z})^-$ for the eigenvalue -1 of ρ^2 acting on $H_1(C(x), \mathbb{Z})$.

(2) ([MT04, Proposition 2.6]) We set

$$\begin{aligned} \mathbf{a}_1 &= A_1, & \mathbf{a}_2 &= A_1 + A_2 + B_2, & \mathbf{a}_3 &= \mathbf{a}_2 + B_3, \\ \mathbf{a}_4 &= \mathbf{a}_3 - A_4 + B_4, & \mathbf{a}_5 &= \mathbf{a}_3 + A_5, & \mathbf{a}_6 &= \mathbf{a}_5 + A_6 + B_6, \end{aligned}$$

and we write $\mathbf{b}_1, \dots, \mathbf{b}_6$ for the cycles given in [MT04, Proposition 2.6]. Following [MT04, Section 2.2], we write $\langle \cdot, \cdot \rangle$ for one half of the intersection form on $H_1(C(x), \mathbb{Z})^-$. The set $\Sigma_1 = \{\mathbf{a}_1, \dots, \mathbf{a}_6, \mathbf{b}_1, \dots, \mathbf{b}_6\}$ is a symplectic basis of a sub-Hodge structure $L_1 \subset H_1(C(x), \mathbb{Z})^-$ that is principally polarized by $\langle \cdot, \cdot \rangle$, and $\langle \mathbf{a}_j, \mathbf{b}_j \rangle = -1$ for $1 \leq j \leq 6$. Moreover, ρ acts on Σ_1 by

$${}^t(\mathbf{a}_1, \dots, \mathbf{a}_6, \mathbf{b}_1, \dots, \mathbf{b}_6) \mapsto \begin{pmatrix} 0 & -U \\ U & 0 \end{pmatrix} {}^t(\mathbf{a}_1, \dots, \mathbf{a}_6, \mathbf{b}_1, \dots, \mathbf{b}_6).$$

(3) ([MT04, Sections 3.1 and 3.2], [MMN07, Section 2.3]) We extend $\langle \cdot, \cdot \rangle$ to a bilinear form on $H_1(C(x), \mathbb{R})^-$, and we give this space the complex structure for which ρ acts as multiplication by $-i$. The form $h(\delta_1, \delta_2) = \langle \delta_1, \rho \delta_2 \rangle - \langle \delta_1, \delta_2 \rangle i$ is a Hermitian form of signature $(5, 1)$, and the period

$$\eta = \left(\int_{\mathbf{a}_1} \frac{dz}{w}, \dots, \int_{\mathbf{a}_6} \frac{dz}{w} \right)$$

satisfies $h_U(\eta, \eta) < 0$.

(4) ([MT04, Lemma 3.6], [MY93, Proposition 3.1], [MMN07, Section 2.3]) The chains of assertion (1) continue along every path in $X(2, 8)$, and the continuation depends only on the homotopy class of the path. The half twist interchanging two branch points x_j and x_k acts on the continued cycles by a unitary reflection g_{jk} of order four, and the full twist acts by the reflection g_{jk}^2 of order two.

By Fact 2.2(2), the matrix U occurs as a block in the matrix of ρ with respect to Σ_1 ; the matrix U is not the matrix of the intersection form on $H_1(C(x), \mathbb{Z})$. Moreover, Fact 2.2(2) gives $h(\mathbf{a}_j, \mathbf{a}_k) = \langle \mathbf{a}_j, \rho \mathbf{a}_k \rangle = U_{jk}$, so the form h is expressed by h_U in the coordinates with respect to the basis $\mathbf{a}_1, \dots, \mathbf{a}_6$ of the complex vector space $H_1(C(x), \mathbb{R})^-$. The period η is the coordinate vector of the linear form $\delta \mapsto \int_{\delta} \frac{dz}{w}$ with respect to the dual basis, and since $U^{-1} = U$, the form induced by h on the dual space is again expressed by h_U in these coordinates. We write $\eta_1, \dots, \eta_6$ for the components of the period η of Fact 2.2(3), so that $\eta = {}^t(\eta_1, \dots, \eta_6)$ with $\eta_j = \int_{\mathbf{a}_j} \frac{dz}{w}$.

Fact 2.2 defines η only on $X_{\mathbb{R}}$, and we extend η to the whole of $X(2, 8)$ by analytic continuation. We fix a base point $\dot{x} \in X_{\mathbb{R}}$. For any $x \in X(2, 8)$, we choose a path from $\dot{x}$ to x , and we define the cycles $\mathbf{a}_1, \dots, \mathbf{a}_6$ on $C(x)$ by continuing the cycles on $C(\dot{x})$ along that path. The integrals $\eta_j = \int_{\mathbf{a}_j} \frac{dz}{w}$ continue with these cycles and

are holomorphic in x , so η is a multivalued holomorphic function on $X(2, 8)$. The continuation preserves the intersection form and commutes with ρ , hence preserves h . The Hodge–Riemann bilinear relations then give the inequality of Fact 2.2(3) at every point of $X(2, 8)$. By Fact 2.2(4), the period η depends only on the homotopy class of the path from $\dot{x}$ to x , and two branches of η differ by the monodromy of the family. We take $\dot{x}$ as the base point of the universal cover $\varpi: \tilde{X}(2, 8) \rightarrow X(2, 8)$, and for $\tilde{x} \in \tilde{X}(2, 8)$ we write $\eta(\tilde{x})$ for the continuation of η along a path representing $\tilde{x}$. We thus obtain a single-valued holomorphic map

$$\tilde{X}(2, 8) \ni \tilde{x} \mapsto \eta(\tilde{x}) \in \mathcal{B}, \quad \mathcal{B} = \{\xi \in \mathbb{C}^6 \mid h_U(\xi, \xi) < 0\}.$$

Since $X_{\mathbb{R}}$ is convex, it is simply connected. Hence the base point determines a unique lift of $X_{\mathbb{R}}$ to $\tilde{X}(2, 8)$. On this lift, the components of $\eta(\tilde{x})$ are the integrals $\eta_j = \int_{\alpha_j} \frac{dz}{w}$ of Fact 2.2, with the cycles and the branch fixed above. We use $X_{\mathbb{R}}$ again in Section 4, where we take the limit of η at the degenerate configuration along the closure of $X_{\mathbb{R}}$.

2.4. The Picard Modular Group and the Period Map

Following [MMN07, Section 2.3], we define the Hermitian matrix H_6 of size 6 by

$$(H_6)_{jj} = 2, \quad (H_6)_{j,j+1} = -1 + i, \quad (H_6)_{j+1,j} = -1 - i, \\ (H_6)_{jk} = 0 \quad (|j - k| \geq 2),$$

and we write $T = N_1 - iN_2$ for the coordinate change given in the same section, where

$$(2.1) \quad N_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The entries of N_1 and N_2 are the coefficients of the cycles of Fact 2.2:

$$\alpha_j = \sum_{k=1}^6 ((N_1)_{jk} A_k + (N_2)_{jk} B_k) \quad (1 \leq j \leq 6).$$

Since $(\rho^2)^* \frac{dz}{w} = -\frac{dz}{w}$ and $\rho^* \frac{dz}{w} = -i \frac{dz}{w}$, we have $\int_{A_k} \frac{dz}{w} = 2\mathcal{I}_k$ and $\int_{B_k} \frac{dz}{w} = -2i\mathcal{I}_k$. Hence we have $\eta = 2T^t(\mathcal{I}_1, \dots, \mathcal{I}_6)$ on the real chamber. A direct computation gives $\det T = 2 + 2i$ and

$$TH_6T^* = 2U, \quad (T^{-1})^* H_6^{-1} T^{-1} = \frac{1}{2} U.$$

Therefore the map $\mathcal{B} \ni \eta \mapsto T^{-1}\eta \in \mathbb{C}^6$ identifies $\mathcal{B}$ with $\{\xi \in \mathbb{C}^6 \mid {}^t \bar{\xi} H_6^{-1} \xi < 0\}$, and hence $\mathbb{B}_5$ with the ball of [MMN07]. We define the Picard modular group and its principal congruence subgroup of level $1 + i$ by

$$\Gamma = \mathrm{U}(H_6, \mathbb{Z}[i]) = \{g \in \mathrm{GL}(6, \mathbb{Z}[i]) \mid gH_6g^* = H_6\}, \\ \Gamma(1 + i) = \ker(\Gamma \longrightarrow \mathrm{GL}(6, \mathbb{Z}[i]/(1 + i))),$$

and we transport them to the coordinates η by putting

$$\Gamma_U = T \Gamma T^{-1}, \quad \Gamma_M = T \Gamma(1 + i) T^{-1}.$$

The groups Γ_U and Γ_M are isomorphic to Γ and $\Gamma(1+i)$, respectively, but distinct from them. Since $\Gamma(1+i)$ is a kernel, it is normal in Γ , and Γ_M is normal in Γ_U . Since $1+i = i(1-i)$, the ideals $(1+i)$ and $(1-i)$ of $\mathbb{Z}[i]$ coincide, and the group $\Gamma(1+i)$ is the group $\Gamma(1-i)$ of [MY93] and [MMN07]. By [MY93, Proposition 3.1 and Theorem 4.1], the group $\Gamma(1+i)$ is generated by 21 of the reflections g_{jk}^2 . The index pairs (j, k) are those listed in [MY93, Proposition 3.1]. A direct computation shows that the corresponding 21 matrices $Tg_{jk}^2T^{-1}$ have entries in $\mathbb{Z}[i]$ and preserve U ; hence Γ_M is a subgroup of $U(U, \mathbb{Z}[i])$. With $T = N_1 - iN_2$ as in (2.1), the congruence condition refers to the coordinates $T^{-1}\eta$ and not to the coordinates η . The elements of Γ_M are not all congruent to I_6 modulo $1+i$; cf. [MT02, Remark 1]. The deck transformations of ϖ act through the monodromy representation

$$\begin{aligned} \mu: \pi_1(X(2, 8), \dot{x}) &\longrightarrow \Gamma_M \subset U(U, \mathbb{Z}[i]), \\ \eta(\gamma \cdot \tilde{x}) &= \mu(\gamma) \eta(\tilde{x}) \quad (\gamma \in \pi_1(X(2, 8), \dot{x})), \end{aligned}$$

and the image of μ is the whole group Γ_M by [MY93, Theorem 4.1]. By the equivariance above, the class of $\eta(\tilde{x})$ in $\Gamma_M \backslash \mathbb{B}_5$ depends only on $\varpi(\tilde{x})$, and we obtain the period map

$$\text{per}: X(2, 8) \longrightarrow \Gamma_M \backslash \mathbb{B}_5, \quad \mathbb{B}_5 = \{\xi \in \mathbb{P}^5 \mid h_U(\xi, \xi) < 0\},$$

from the configuration space $X(2, 8)$ to the ball quotient. Composing the map $\tilde{x} \mapsto \eta(\tilde{x})$ with the projection $\mathcal{B} \rightarrow \mathbb{B}_5$, we obtain a lift $\widetilde{\text{per}}: \tilde{X}(2, 8) \rightarrow \mathbb{B}_5$ of per . The map $\tilde{x} \mapsto \eta(\tilde{x})$ is determined by the differential dz/w and the homology marking fixed above, and it fixes the scale of $\eta(\tilde{x})$ as well as the class $\widetilde{\text{per}}(\tilde{x})$ of $\eta(\tilde{x})$ in $\mathbb{B}_5$. If F is a holomorphic function on $\mathcal{B}$ invariant under Γ_M , then the function $\tilde{x} \mapsto F(\eta(\tilde{x}))$ on $\tilde{X}(2, 8)$ is invariant under the deck transformations and descends to a holomorphic function on $X(2, 8)$.

2.5. Theta Functions

We write $\mathfrak{S}_n$ for the Siegel upper half-space of degree n , that is, the set of symmetric matrices τ of size n over $\mathbb{C}$ whose imaginary part is positive definite. We set $\mathbf{e}(t) = \exp(2\pi it)$. For $n = 1$, we have $\mathfrak{S}_1 = \mathbb{H}$. For a subring $\mathcal{R}$ of $\mathbb{C}$, we set

$$\text{Sp}(2n, \mathcal{R}) = \{M \in \text{GL}(2n, \mathcal{R}) \mid MJ_{2n} {}^t M = J_{2n}\}, \quad J_{2n} = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix},$$

where I_n and O_n are the identity and zero matrices of size n , respectively. Thus $\text{Sp}(2n, \mathcal{R})$ consists of matrices of size $2n$. The group $\text{Sp}(2n, \mathbb{R})$ acts on $\mathfrak{S}_n$ by $M \cdot \tau = (M_{11}\tau + M_{12})(M_{21}\tau + M_{22})^{-1}$, where we write M in $n \times n$ blocks as $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$. We define Riemann's theta function with half characteristics $(a/2, b/2)$ in the variables $(\zeta, \tau) \in \mathbb{C}^n \times \mathfrak{S}_n$ by the series

$$(2.2) \quad \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\zeta, \tau) = \sum_{k \in \mathbb{Z}^n} \mathbf{e} \left(\frac{1}{2} \left(k + \frac{1}{2}a \right) \tau \left(k + \frac{1}{2}a \right) + \left(k + \frac{1}{2}a \right) \left(\zeta + \frac{1}{2}b \right) \right),$$

where $a, b \in \mathbb{Z}^n$. The series converges absolutely and uniformly on any compact set in $\mathbb{C}^n \times \mathfrak{S}_n$. The same series converges for arbitrary real characteristics, and we use it with half-integral b in Proposition 4.3. Note that the characteristics enter (2.2) as $a/2$ and $b/2$. Thus, in $\vartheta \begin{bmatrix} 110000 \\ 001100 \end{bmatrix}$, we have $a = (1, 1, 0, 0, 0, 0)$ and

$b = (0, 0, 1, 1, 0, 0)$. For $n = 1$ and real a and b , we write $\vartheta_{ab}(\tau) = \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (0, \tau)$, and we separate the two subscripts by a comma when either is a compound expression. For $a, b \in \{0, 1\}$, the functions ϑ_{ab} are Jacobi's theta constants.

2.6. The Automorphic Forms

Following [MMN07, Section 2.3], we embed the complex ball $\mathbb{B}_5$ into $\mathfrak{S}_6$ by

$$\iota(\eta) = i \left(U - \frac{2}{{}^t\eta U \eta} \eta {}^t\eta \right),$$

and we write $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\eta) = \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(0, \iota(\eta))$ for the theta constants with characteristics $a, b \in \mathbb{Z}^6$. Then we have the following fact.

- Fact 2.3.** (1) ([MMN07, Section 2.3]) *The map ι is well defined on $\mathcal{B}$ with values in $\mathfrak{S}_6$, and $\iota(\eta)$ depends only on the class of η in $\mathbb{B}_5$.*
(2) ([MMN07, Section 2.3]) *For $g \in \Gamma_U$, we write $g = A + iBU$ with real matrices A, B . The assignment*

$$j(g) = \begin{pmatrix} A & B \\ -UBU & UAU \end{pmatrix}$$

defines a homomorphism into $\mathrm{Sp}(12, \mathbb{Q})$ satisfying $\iota(g\eta) = j(g) \cdot \iota(\eta)$.

- (3) ([MT04, Section 3.2], [MT02, Proposition 6], [MMN07, Section 2.3]) *For the period η , the matrix $\iota(\eta)$ is the normalized period matrix of the principally polarized abelian variety attached to (L_1, Σ_1) .*

By Fact 2.3(3), the results of [MT04] and [MMN07] hold at $\iota(\eta)$, so that the values $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\eta)$ are the theta constants of that abelian variety.

Remark 2.4. We follow the convention of [MT04] and [MT02]: the branches of w fixed above are obtained by analytic continuation through $\mathbb{H}$. In [MT02, Section 2], where all eight branch points are taken to be finite, the lift of $[x_j, x_{j+1}]$ is specified by the condition $\exp(\frac{\pi i}{4}j)w \in [0, \infty)$.

The forms f_J of [MMN07, Section 5] that we use are attached to the five double coset representatives

$$J_1 = \langle 12; 34; 56; 78 \rangle, \quad J_2 = \langle 13; 24; 56; 78 \rangle, \quad J_3 = \langle 13; 25; 46; 78 \rangle, \\ J_4 = \langle 13; 24; 57; 68 \rangle, \quad J_5 = \langle 13; 25; 47; 68 \rangle$$

of $\mathrm{Stab}(J_1) \backslash S_8 / \mathrm{Stab}(J_1)$, where $\mathrm{Stab}(J_1)$ is the stabilizer of J_1 in S_8 . These five forms are given by

$$\begin{aligned} f_{J_1} &= \vartheta \begin{bmatrix} 000000 \\ 000000 \end{bmatrix} \vartheta \begin{bmatrix} 001111 \\ 001111 \end{bmatrix} \vartheta \begin{bmatrix} 110011 \\ 110011 \end{bmatrix} \vartheta \begin{bmatrix} 111100 \\ 111100 \end{bmatrix}, \\ f_{J_2} &= 4 \vartheta \begin{bmatrix} 110000 \\ 000000 \end{bmatrix} \vartheta \begin{bmatrix} 111111 \\ 001111 \end{bmatrix} \vartheta \begin{bmatrix} 000011 \\ 110011 \end{bmatrix} \vartheta \begin{bmatrix} 001100 \\ 111100 \end{bmatrix}, \\ (2.3) \quad f_{J_3} &= \left(\vartheta \begin{bmatrix} 111100 \\ 000000 \end{bmatrix}^2 - \vartheta \begin{bmatrix} 110000 \\ 001100 \end{bmatrix}^2 \right) \left(\vartheta \begin{bmatrix} 000011 \\ 111111 \end{bmatrix}^2 - \vartheta \begin{bmatrix} 001111 \\ 110011 \end{bmatrix}^2 \right), \end{aligned}$$

$$f_{J_4} = \left(\vartheta \begin{bmatrix} 110011 \\ 000000 \end{bmatrix}^2 - \vartheta \begin{bmatrix} 110000 \\ 000011 \end{bmatrix}^2 \right) \left(\vartheta \begin{bmatrix} 001100 \\ 111111 \end{bmatrix}^2 - \vartheta \begin{bmatrix} 001111 \\ 111100 \end{bmatrix}^2 \right),$$

$$f_{J_5} = \frac{1}{4} \prod_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \left(\vartheta \begin{bmatrix} 111111 \\ 000000 \end{bmatrix} + \epsilon_1 \vartheta \begin{bmatrix} 110000 \\ 001111 \end{bmatrix} \right. \\ \left. + \epsilon_2 \vartheta \begin{bmatrix} 001100 \\ 110011 \end{bmatrix} + \epsilon_1 \epsilon_2 \vartheta \begin{bmatrix} 000011 \\ 111100 \end{bmatrix} \right)$$

(see [MMN07, Fact 3 and Propositions 4–7]). For $k = 1, \dots, 5$, the sign ε_{J_k} is $+1$; hence $\hat{x}_{J_k} = x_{J_k}$. In the normalization of [MMN07, Section 5], the forms f_J represent, as explicit polynomials in theta constants, the forms $T_J^{(2)}$ constructed in [MT04]. By [MMN07, Fact 5], which rests on [MT04, Theorem 6.4], the collection $(f_J)_J$ is equivariant under S_8 , and for every $\tilde{x} \in \tilde{X}(2, 8)$ the projective points $[\dots : f_J(\eta) : \dots]_J$ and $[\dots : \hat{x}_J(x) : \dots]_J$ coincide, where $x = \varpi(\tilde{x})$ and $\eta = \eta(\tilde{x})$.

The following proposition records the automorphy of the forms f_J .

Proposition 2.5. *For every $g \in \Gamma_M$, every $(2, 2, 2, 2)$ -partition J , and every $\eta \in \mathcal{B}$, we have*

$$(2.4) \quad f_J(g\eta) = \chi(j(g), \iota(\eta))^2 f_J(\eta),$$

$$(2.5) \quad q_U(g\eta) = \det(g) \chi(j(g), \iota(\eta)) q_U(\eta),$$

where j is the homomorphism of Fact 2.3(2) and $\chi(M, \tau) = \det(M_{21}\tau + M_{22})$ for $M \in \mathrm{Sp}(12, \mathbb{Q})$ and $\tau \in \mathfrak{S}_6$. In particular, we have

$$\frac{f_J(g\eta)}{q_U(g\eta)^2} = \frac{f_J(\eta)}{q_U(\eta)^2}.$$

PROOF. The forms f_J are defined in [MMN07, Section 2.4, Facts 3–5] as the translates of f_{J_1} under Γ_U , and the same facts give

$$f_J(g\eta) = \psi(g, \eta)^2 f_J(\eta), \quad \psi(g, \eta) = \frac{q_U(g\eta)}{q_U(\eta)},$$

for $g \in \Gamma_M$. The quotient $\psi(g, \eta)$ is well defined by Lemma 2.1. This transformation law applies to each form f_J as a whole, and not to the individual theta constants occurring in (2.3). By [MMN07, Lemma 1], we have $\psi(g, \eta) = \det(g) \chi(j(g), \iota(\eta))$, which gives (2.5). Moreover, [MMN07, Proposition 2] gives $\det(g)^2 = \mathrm{sign}(s(g)) = 1$ for $g \in \Gamma_M = \ker s$, where $s: \Gamma_U \rightarrow S_8$ is the homomorphism of [MMN07, Section 2.3] sending $Tg_{jk}T^{-1}$ to the transposition (j, k) ; the equality $\det(g)^2 = 1$ yields (2.4). Dividing (2.4) by the square of (2.5) and using $\det(g)^2 = 1$, we obtain the last assertion. $\square$

3. CONSTANCY OF THE RATIO

In this section, we show that the ratio $f_J(\eta)/(q_U(\eta)^2 \hat{x}_J(x))$ descends to a function on $X(2, 8)$ that is constant and independent of J .

Throughout this section, we regard $X(2, 8)$ as a Zariski open subset of $\mathbb{P}^5$ via the coordinates $(x_2, \dots, x_6)$. We use the homogeneous coordinates $[y_1 : y_2 : \dots : y_6]$ on $\mathbb{P}^5$, so that $x_j = y_j/y_1$. We write y for a point of $\mathbb{P}^5$. The boundary $\mathbb{P}^5 \setminus X(2, 8)$ is

the union of the 21 hyperplanes

$$(3.1) \quad \begin{aligned} D_{jk} &= \{y \in \mathbb{P}^5 \mid y_j = y_k\} \quad (2 \leq j < k \leq 6), \\ D_{0j} &= \{y \in \mathbb{P}^5 \mid y_j = 0\}, \quad D_{1j} = \{y \in \mathbb{P}^5 \mid y_j = y_1\} \quad (2 \leq j \leq 6), \\ H_\infty &= \{y \in \mathbb{P}^5 \mid y_1 = 0\}, \end{aligned}$$

that is, of the $\binom{5}{2} + 5 + 5 = 20$ hyperplanes $D_{\bullet\bullet}$ and the hyperplane at infinity. By a generic point of a boundary hyperplane, we mean a point lying outside the intersections of that hyperplane with the other boundary hyperplanes. A generic point of each of the 20 hyperplanes $D_{\bullet\bullet}$ corresponds to the confluence of exactly two of the eight points, under the normalization $x_1 = 0$, $x_7 = 1$, and $x_8 = \infty$. In the coordinates $(x_2, \dots, x_6)$, H_∞ is not given by any of the affine equations $x_j = x_k$, $x_j = 0$, or $x_j = 1$. A generic point of H_∞ corresponds to such a confluence only after a projective change of coordinates; see Step 3' in the proof of Proposition 3.4. The locus where more than two of the eight points coincide has codimension at least 2 in $\mathbb{P}^5$, and so does the locus where each of two disjoint pairs of the eight points coincides.

We recall Deligne–Mostow’s extension results, which apply near a generic point of each of the 20 hyperplanes $D_{\bullet\bullet}$.

Fact 3.1. ([DM86, Corollary 6.8, Proposition 8.7, and Sections 9.5–9.6]) *After we pass to a finite local cover and fix the marking, the period η extends holomorphically across each of the 20 hyperplanes $D_{\bullet\bullet}$ in (3.1) near a generic point of that hyperplane. The extension takes values in $\mathcal{B}$.*

We next separate the covariance under changes of coordinates on $\mathbb{P}^1$ from the equivariance under changes of the marking; the following lemma gives the covariance.

Lemma 3.2. *Let $p_1, \dots, p_8$ be pairwise distinct points of $\mathbb{P}^1$, written $p_l = [u_l : v_l]$, and put*

$$[p_l, p_m] = u_l v_m - u_m v_l, \quad P_J = \prod_{k=1}^4 [p_{j_{2k-1}}, p_{j_{2k}}], \quad \omega = \frac{v du - u dv}{w},$$

where $p = [u : v]$ and $w^4 = \prod_{l=1}^8 [p, p_l]$. Here η denotes the period of ω with respect to the cycles $\mathbf{a}_1, \dots, \mathbf{a}_6$ of Fact 2.2. We transport these cycles by the projective change of coordinates that carries the normalized representative of the class of $(p_1, \dots, p_8)$ to $(p_1, \dots, p_8)$. We define

$$G_J(p_1, \dots, p_8) = \frac{f_J(\eta)}{q_U(\eta)^2 \varepsilon_J P_J}.$$

Then G_J is independent of the lifts (u_l, v_l) of the points p_l and invariant under $\mathrm{PGL}(2, \mathbb{C})$. In the affine chart $p_l = [x_l : 1]$, $p_8 = [1 : 0]$, this definition gives

$$G_J = \frac{f_J(\eta)}{q_U(\eta)^2 \hat{x}_J(x)}.$$

PROOF. Suppose that p_l is replaced by $t_l p_l$ with $t_l \in \mathbb{C}^\times$ for $1 \leq l \leq 8$, and put $R = \prod_{l=1}^8 t_l$. Then P_J is multiplied by R , independently of J . We choose a local fourth root of R . Then w is multiplied by $R^{1/4}$, so ω and η are multiplied by $R^{-1/4}$, and $q_U(\eta)^2$ is multiplied by R^{-1} . By Fact 2.3(1), the matrix $\iota(\eta)$, and therefore

$f_J(\eta)$, is unchanged by scalar multiplication of η . Thus G_J is independent of the lifts. A different choice of the fourth root only multiplies η by a fourth root of unity and yields the same conclusion.

For $A \in \mathrm{GL}(2, \mathbb{C})$, we replace p_l by Ap_l for $1 \leq l \leq 8$, and we have

$$[Ap_l, Ap_m] = \det(A)[p_l, p_m],$$

so that P_J is multiplied by $\det(A)^4$. The numerator $v du - u dv$ is multiplied by $\det(A)$, and the product $\prod_{l=1}^8 [p, p_l]$ is multiplied by $\det(A)^8$. Therefore ω and η are multiplied by $\det(A)^{-1}$, and $q_U(\eta)^2$ is multiplied by $\det(A)^{-4}$. Thus the two factors in the denominator of G_J are multiplied by $\det(A)^{-4}$ and $\det(A)^4$, respectively, but $f_J(\eta)$ is unchanged. Hence G_J is invariant under $\mathrm{PGL}(2, \mathbb{C})$.

In the affine chart of the statement, we write w_h for the function of the homogeneous model, whose equation is

$$w_h^4 = - \prod_{j=1}^7 (z - x_j).$$

The affine model used in Section 2 is $w_a^4 = \prod_{j=1}^7 (z - x_j)$. We choose a primitive eighth root of unity ζ_8 , so that $\zeta_8^4 = -1$, and identify the two models by $w_h = \zeta_8 w_a$. Then $\eta_h = \zeta_8^{-1} \eta_a$ and

$$q_U(\eta_h)^2 = \zeta_8^{-4} q_U(\eta_a)^2 = -q_U(\eta_a)^2.$$

Since $P_J = -x_J$ and $f_J(\eta)$ is unchanged by scalar multiplication of η , the two negative signs cancel and give the affine formula in the statement. $\square$

The following lemma gives the equivariance under changes of the marking.

Lemma 3.3. *Let $g \in \Gamma_U$, and put $\sigma = s(g) \in S_8$, where $s: \Gamma_U \rightarrow S_8$ is the homomorphism of [MMN07, Section 2.3]. Then, for every $(2, 2, 2, 2)$ -partition J , there is a sign $\varepsilon(\sigma, J) \in \{\pm 1\}$ such that*

$$\hat{x}_J(\sigma \cdot x) = \varepsilon(\sigma, J) \hat{x}_{\sigma^{-1}(J)}(x), \quad f_J(g\eta) = \varepsilon(\sigma, J) \psi(g, \eta)^2 f_{\sigma^{-1}(J)}(\eta),$$

where $\psi(g, \eta) = q_U(g\eta)/q_U(\eta)$. In particular, we have

$$\frac{f_J(g\eta)}{q_U(g\eta)^2 \hat{x}_J(\sigma \cdot x)} = \frac{f_{\sigma^{-1}(J)}(\eta)}{q_U(\eta)^2 \hat{x}_{\sigma^{-1}(J)}(x)}.$$

PROOF. Reordering the two entries in each factor of $x_J(\sigma \cdot x) = \pm x_{\sigma^{-1}(J)}(x)$ and inserting the signs ε_J and $\varepsilon_{\sigma^{-1}(J)}$ of $\hat{x}_J = \varepsilon_J x_J$, we obtain the first identity, with a sign $\varepsilon(\sigma, J)$ depending only on σ and J . For the second identity, we use the action $f^g(\eta) = \psi(g, \eta)^{-2} f(g\eta)$ of Γ_U on the automorphic forms of weight 2 with respect to Γ_M ; cf. [MMN07, Section 2.4]. By [MMN07, Fact 5] and the normalization of [MMN07, Section 5], the collections $(f_J)_J$ and $(\hat{x}_J)_J$ are equivariant under S_8 with the same multipliers; that is, $f_J^g = \varepsilon(\sigma, J) f_{\sigma^{-1}(J)}$ with the sign $\varepsilon(\sigma, J)$ of the first identity. This equality is the second identity. Since every coordinate $\hat{x}_J(x)$ is nonzero on $X(2, 8)$, the equality of the two projective points established in [MMN07, Fact 5] shows that every coordinate $f_J(\eta)$ is nonzero, and the quotients in the last assertion are defined. Dividing the second identity by $q_U(g\eta)^2 = \psi(g, \eta)^2 q_U(\eta)^2$ and by the first identity, we obtain the last assertion. $\square$

Then we have the following proposition.

Proposition 3.4. *For each $(2, 2, 2, 2)$ -partition J , the function G_J of Lemma 3.2 descends from $\tilde{X}(2, 8)$ to a single-valued holomorphic function on $X(2, 8)$ and is constant. Moreover, the constant is independent of J ; we write G for the common function.*

PROOF. *Step 1: single-valuedness.* For $\xi \in \mathcal{B}$, we set $F_J(\xi) = f_J(\xi)/q_U(\xi)^2$. By Lemma 2.1, the denominator does not vanish, so F_J is holomorphic. Proposition 2.5 establishes the cancellation of the factor $\psi(g, \eta)^2$ and shows that F_J is invariant under Γ_M . Thus the function $\hat{x} \mapsto F_J(\eta(\hat{x}))$, and therefore G_J , is invariant under the deck transformations of ϖ and descends to a single-valued holomorphic function on $X(2, 8)$.

Step 2: independence of J . As recalled in Section 2, the forms f_J in (2.3) represent the forms $T_J^{(2)}$ of [MT04] in the normalization of [MMN07, Section 5]. By [MMN07, Fact 5], which rests on [MT04, Theorem 6.4], the projective points $[\dots : f_J(\eta) : \dots]_J$ and $[\dots : \hat{x}_J(x) : \dots]_J$ coincide. Since every coordinate $\hat{x}_J(x)$ is nonzero on $X(2, 8)$, the equality of the two projective points shows that the ratio $f_J(\eta)/\hat{x}_J(x)$ is independent of J . Dividing the ratio $f_J(\eta)/\hat{x}_J(x)$ by the common factor $q_U(\eta)^2$, which is nonzero by Lemma 2.1, we obtain $G_J = G_{J'}$ on $X(2, 8)$ for all J and J' .

Step 3: extension across the hyperplanes $D_{\bullet\bullet}$. Let D be one of the 20 hyperplanes $D_{\bullet\bullet}$ in (3.1), and let x_0 be a generic point of D . We choose a partition J' that does not pair the two points coinciding on D , so that $\hat{x}_{J'}(x_0) \neq 0$. By Fact 3.1, the period η extends holomorphically across the preimage of D , with values in $\mathcal{B}$, after we pass to a finite local cover and fix the marking. Since $F_{J'}$ is holomorphic on $\mathcal{B}$ and $\hat{x}_{J'}(x_0) \neq 0$, the function $F_{J'}(\eta)/\hat{x}_{J'}$ extends holomorphically to this local cover. By Step 1, away from the preimage of D , the extension of $F_{J'}(\eta)/\hat{x}_{J'}$ is invariant under the deck transformations of this local cover, and hence the extension is invariant everywhere by the identity theorem. The extension of $F_{J'}(\eta)/\hat{x}_{J'}$ therefore descends to a holomorphic extension of G across D at x_0 .

Step 3': extension across the hyperplane at infinity. We change the coordinate on $\mathbb{P}^1$ by $\mathbb{P}^1 \ni z \mapsto y_1 z / y_2 \in \mathbb{P}^1$ and then transpose the labels 2 and 7. The composite takes the configuration $(0, x_2, \dots, x_6, 1, \infty)$ with $x_j = y_j / y_1$ to the configuration $(0, x'_2, \dots, x'_6, 1, \infty)$ with

$$x'_2 = \frac{y_1}{y_2}, \quad x'_j = \frac{y_j}{y_2} \quad (3 \leq j \leq 6).$$

In the homogeneous coordinates of (3.1), the composite is the linear map

$$\sigma_\infty : \mathbb{P}^5 \ni [y_1 : y_2 : y_3 : y_4 : y_5 : y_6] \mapsto [y_2 : y_1 : y_3 : y_4 : y_5 : y_6] \in \mathbb{P}^5,$$

an involutive automorphism of $\mathbb{P}^5$ that carries H_∞ onto D_{02} . The map σ_∞ carries a generic point of H_∞ to a generic point of D_{02} . Moreover, σ_∞ restricts to a biholomorphism from a neighborhood of that generic point of H_∞ onto a neighborhood of the image of that point. The transposition of the labels 2 and 7 is induced by the unitary reflection g_{27} of Fact 2.2, and $Tg_{27}T^{-1}$ lies in Γ_U and satisfies $s(Tg_{27}T^{-1}) = (27)$. Hence, by Lemmas 3.2 and 3.3, together with the equality $G_J = G_{J'}$ established in Step 2, we have $G \circ \sigma_\infty = G$. By Step 3, G extends holomorphically across D_{02} at a generic point, so G extends holomorphically across H_∞ at a generic point as well.

Step 4: extension to $\mathbb{P}^5$ and constancy. We write A for the union of the pairwise intersections of the 21 boundary hyperplanes in (3.1). The set A is a finite union of

linear subspaces of codimension 2 in $\mathbb{P}^5$; in particular, A is a closed analytic subset of codimension 2. Every point of $\mathbb{P}^5 \setminus A$ lies in $X(2, 8)$ or is a generic point of exactly one boundary hyperplane. By Steps 1, 3, and 3', G extends holomorphically across every boundary hyperplane at every generic point of that hyperplane. Each local extension is unique because it agrees with G on the dense open intersection of its domain with $X(2, 8)$. Therefore the identity theorem shows that these extensions glue to a holomorphic function on $\mathbb{P}^5 \setminus A$. By Hartogs' theorem, G extends holomorphically to $\mathbb{P}^5$. Since $\mathbb{P}^5$ is compact and connected, every global holomorphic function on it is constant. Hence G is constant. $\square$

Remark 3.5. If we change the lift $\tilde{x}$ by a deck transformation, then η changes by an element of Γ_M . By Proposition 3.4, this change does not affect the Main Theorem. A change of the marking by an element of Γ_U can also permute the forms f_J and the polynomials $\hat{x}_J$. By Lemma 3.3, the collection of identities (1.1) is preserved under such a change.

By Proposition 3.4, the function G is a constant, and we write κ for its value. We have only to evaluate G at one degenerate configuration. We use the auxiliary partition

$$J^* = \langle 14; 25; 36; 78 \rangle, \quad x_{J^*} = (x_1 - x_4)(x_2 - x_5)(x_3 - x_6).$$

The corresponding automorphic form is

$$(3.2) \quad f_{J^*} = \left(\vartheta \begin{bmatrix} 111100 \\ 000000 \end{bmatrix}^2 + \vartheta \begin{bmatrix} 110000 \\ 001100 \end{bmatrix}^2 \right) \left(\vartheta \begin{bmatrix} 000011 \\ 111111 \end{bmatrix}^2 + \vartheta \begin{bmatrix} 001111 \\ 110011 \end{bmatrix}^2 \right)$$

by entry #15 of the table in [MMN07, Theorem 2]. In that table, f_{J^*} is the entry labeled g_3 with $\varepsilon = \varepsilon' = +1$. It is attached to $+x_{\langle 14; 25; 36; 78 \rangle}$ and has the same four characteristics as f_{J_3} ; hence $\hat{x}_{J^*} = x_{J^*}$.

4. DEGENERATION AND SPLITTING OF THE THETA CONSTANTS

In this section, we compute the period η at a degenerate configuration, express the constant κ in terms of the value of f_{J^*} there, and split the theta constants into theta constants on $\mathbb{H}$.

4.1. The Degenerate Configuration

We fix $c \in (0, 1)$ and consider the degenerate configuration

$$(0, 0, 0, c, c, c, 1, \infty),$$

the limit of $x \in X(2, 8)$ as $x_2, x_3 \rightarrow 0$ and $x_4, x_5, x_6 \rightarrow c$. The normalization of the limit curve $w^4 = z^3(z - c)^3(z - 1)$, that is, the smooth projective model of this curve, has genus 3. In fact, the map from the normalization to $\mathbb{P}^1$ is of degree four and totally ramified over $0, c, 1, \infty$, so the Riemann–Hurwitz formula gives $2g - 2 = 4(-2) + 4(4 - 1) = 4$ for the genus g of the normalization, and hence $g = 3$. Each factor $z - x_j$ contributes the exponent $\frac{1}{4}$ to $\frac{dz}{w}$, so each of the two triples of coinciding branch points contributes the total exponent $3 \cdot \frac{1}{4} = \frac{3}{4} < 1$; hence all period integrals converge. Therefore η extends continuously to the degenerate configuration along the closure of $X_{\mathbb{R}}$, and we write η_c for the limit value. We use only this one-sided continuity here, since Proposition 3.4 shows that the function G is constant on $X(2, 8)$.

Then we have the following proposition.

Proposition 4.1. *We have*

$$(4.1) \quad \eta_c = {}^t(0, 0, v, v, v, u), \quad v = 2i\mathcal{K}_1, \quad u = 2i(\mathcal{K}_1 - \sqrt{2}\mathcal{K}_2),$$

where

$$(4.2) \quad \begin{aligned} \mathcal{K}_1 &= \mathcal{K}_1(c) = \int_0^c \frac{dz}{(z^3(c-z)^3(1-z))^{1/4}} = \frac{\Gamma(1/4)^2}{\sqrt{\pi c}} F\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; c\right), \\ \mathcal{K}_2 &= \mathcal{K}_2(c) = \int_c^1 \frac{dz}{(z^3(z-c)^3(1-z))^{1/4}} = \sqrt{2}\pi F\left(\frac{3}{4}, \frac{3}{4}; 1; 1-c\right). \end{aligned}$$

Here F denotes the Gauss hypergeometric series. In both integrands we take the positive real fourth root. In particular, we have

$${}^t\eta_c U \eta_c = u^2 - v^2 = 8\mathcal{K}_2(\sqrt{2}\mathcal{K}_1 - \mathcal{K}_2) = -h_U(\eta_c, \eta_c).$$

PROOF. In the limit, the intervals $\ell_1, \ell_2, \ell_4, \ell_5$ shrink to points and $\mathcal{I}_1 = \mathcal{I}_2 = \mathcal{I}_4 = \mathcal{I}_5 = 0$. In the same limit, the branch of w fixed in Section 2 gives $\mathcal{I}_3 = -\mathcal{K}_1$ on $\ell_3 = [0, c]$ and $\mathcal{I}_6 = \exp(-\pi i/4)\mathcal{K}_2$ on $\ell_6 = [c, 1]$. As in Section 2, the chains of Fact 2.2 satisfy $\int_{A_k} \frac{dz}{w} = 2\mathcal{I}_k$ and $\int_{B_k} \frac{dz}{w} = -2i\mathcal{I}_k$. Thus the cycles of Fact 2.2 give $\eta_1 = \eta_2 = 0$, $\eta_3 = \eta_4 = \eta_5 = -2i\mathcal{I}_3 = v$, and $\eta_6 = -2i\mathcal{I}_3 + 2(1-i)\mathcal{I}_6 = u$. The expressions (4.2) follow from the Euler integral representation of the hypergeometric series, together with $\Gamma(3/4)\Gamma(1/4) = \sqrt{2}\pi$. Since u and v are purely imaginary, we have ${}^t\eta_c U \eta_c = -h_U(\eta_c, \eta_c)$, and (4.1) gives the remaining equalities in the statement. $\square$

We set

$$m = \frac{1 - \sqrt{c}}{2} \in (0, \frac{1}{2}), \quad \mathcal{F}(z) = F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right).$$

Then we have the following lemma.

Lemma 4.2. *We have*

$$(4.3) \quad \mathcal{K}_2 = \frac{\sqrt{2}\pi}{\sqrt{c}} \mathcal{F}(m), \quad \sqrt{2}\mathcal{K}_1 = \frac{\sqrt{2}\pi}{\sqrt{c}} (\mathcal{F}(m) + \mathcal{F}(1-m)).$$

In particular, $\sqrt{2}\mathcal{K}_1 > \mathcal{K}_2$ and ${}^t\eta_c U \eta_c > 0$.

PROOF. We use Euler's transformation and the quadratic transformation given in [EMOT53, Section 2.9, formula (2), and Section 2.11, formula (2)]. We apply Euler's transformation with parameters $(\frac{3}{4}, \frac{3}{4}; 1)$ and the quadratic transformation with $a = b = \frac{1}{4}$, so that $a + b + \frac{1}{2} = 1$. Using $1 - c = 4m(1 - m)$ and the positive real branches, we obtain

$$F\left(\frac{3}{4}, \frac{3}{4}; 1; 1-c\right) = \frac{\mathcal{F}(m)}{1-2m} = \frac{\mathcal{F}(m)}{\sqrt{c}}.$$

The expression for $\mathcal{K}_2$ in (4.2) gives the first equality in (4.3).

We also use the connection formula

$$(4.4) \quad F\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; (1-2m)^2\right) = \frac{\pi^{3/2}}{\Gamma(1/4)^2} (\mathcal{F}(m) + \mathcal{F}(1-m)), \quad 0 < m < 1.$$

To verify (4.4), we use the identity

$$\mathcal{F}\left(\frac{1-z}{2}\right) = \frac{\sqrt{\pi}}{\Gamma(3/4)^2} F\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; z^2\right) - \frac{2\sqrt{\pi}}{\Gamma(1/4)^2} z F\left(\frac{3}{4}, \frac{3}{4}; \frac{3}{2}; z^2\right),$$

which follows from [EMOT53, Section 2.11, formula (3)] by setting $a = b = \frac{1}{4}$ and replacing z by $-z$. We apply this identity at $z = 1 - 2m$ and at $z = 2m - 1$, and add the resulting identities. For $-1 < z < 1$, all branches in this identity are obtained by analytic continuation of the defining series along the real interval from $z = 0$, and none of their arguments crosses the standard cut $[1, \infty)$. The terms that are odd in z cancel, and the identity $\Gamma(3/4)\Gamma(1/4) = \sqrt{2}\pi$ gives (4.4). Substituting (4.4) into the expression for $\mathcal{K}_1$ in (4.2), we obtain the second equality in (4.3). Since $\mathcal{F}(m)$ and $\mathcal{F}(1-m)$ are positive, we obtain $\mathcal{K}_2 > 0$ and $\sqrt{2}\mathcal{K}_1 > \mathcal{K}_2$ from (4.3). The identity ${}^t\eta_c U \eta_c = 8\mathcal{K}_2(\sqrt{2}\mathcal{K}_1 - \mathcal{K}_2)$ of Proposition 4.1 then gives ${}^t\eta_c U \eta_c > 0$. $\square$

4.2. Reduction to a Single Value

By Lemma 4.2 and Fact 2.3(1), the matrix $\iota(\eta_c)$ lies in $\mathfrak{S}_6$, and the theta constants and f_{J^*} also extend continuously to the degenerate configuration. Since the first two coordinates of η_c vanish, a direct computation from the definition of ι shows that $\iota(\eta_c)$ is block diagonal:

$$\iota(\eta_c) = iI_2 \oplus \tau_c, \quad \tau_c = -\frac{i}{u^2 - v^2} \begin{pmatrix} 2v^2 & u^2 + v^2 & 2v^2 & 2uv \\ u^2 + v^2 & 2v^2 & 2v^2 & 2uv \\ 2v^2 & 2v^2 & 3v^2 - u^2 & 2uv \\ 2uv & 2uv & 2uv & u^2 + v^2 \end{pmatrix}.$$

Since $\iota(\eta_c)$ lies in $\mathfrak{S}_6$ and is block diagonal, each of its diagonal blocks has positive definite imaginary part. In particular, the block τ_c lies in $\mathfrak{S}_4$. Every factor of G therefore extends continuously to the degenerate configuration, and $x_{J^*}(0, 0, 0, c, c, 1, \infty) = (0 - c)^3 = -c^3 \neq 0$. Letting x tend to this configuration in the identity $G = \kappa$ of Section 3, we obtain

$$(4.5) \quad \kappa = \frac{f_{J^*}(\eta_c)}{({}^t\eta_c U \eta_c)^2 (-c^3)}.$$

In the rest of this section and in Section 5, we compute the numerator and the denominator of (4.5).

4.3. Splitting of the Theta Constants

The defining series of each theta constant splits along the block decomposition $\iota(\eta_c) = iI_2 \oplus \tau_c$. For every characteristic, writing $a = (a_1, a_2, a')$ and $b = (b_1, b_2, b')$ with $a', b' \in \mathbb{Z}^4$, we have

$$(4.6) \quad \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(\eta_c) = \vartheta_{a_1 b_1}(i) \vartheta_{a_2 b_2}(i) \vartheta \begin{bmatrix} a' \\ b' \end{bmatrix}(\tau_c).$$

Applying (4.6) to the four characteristics of (3.2) and using $\vartheta_{10}(i)^2 = \vartheta_{01}(i)^2 = \vartheta_{00}(i)^2 / \sqrt{2}$, which follows from (4.12) below, we obtain

$$(4.7) \quad f_{J^*}(\eta_c) = \frac{\vartheta_{00}(i)^8}{4} (\Theta_1^2 + \Theta_2^2) (\Theta_3^2 + \Theta_4^2),$$

where $\Theta_1 = \vartheta \begin{bmatrix} 1100 \\ 0000 \end{bmatrix}(\tau_c)$, $\Theta_2 = \vartheta \begin{bmatrix} 0000 \\ 1100 \end{bmatrix}(\tau_c)$, $\Theta_3 = \vartheta \begin{bmatrix} 0011 \\ 1111 \end{bmatrix}(\tau_c)$, and $\Theta_4 = \vartheta \begin{bmatrix} 1111 \\ 0011 \end{bmatrix}(\tau_c)$.

We set $\tau^* = -i(u+v)/(u-v)$, with u and v as in Proposition 4.1. By Lemma 4.2, we have $\tau^*, -1/\tau^* \in i\mathbb{R}_{>0}$. We use these positive imaginary values to fix the branches of the square roots in (4.11), as stated after (4.13) below.

Then we have the following proposition.

Proposition 4.3. *For $n = (n_1, n_2, n_3, n_4) \in \mathbb{Z}^4$, we set*

$$r = (r_1, r_2, r_3, r_4) = n {}^t L, \quad L = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad \det L = -4.$$

Then $n \mapsto r$ is a bijection of $\mathbb{Z}^4$ onto the sublattice of index 4

$$\{r \in \mathbb{Z}^4 \mid r_1 \equiv r_2, r_3 \equiv r_4 \pmod{2}\},$$

with inverse

$$\begin{aligned} n_1 &= \frac{r_1 + r_2 - r_3 + r_4}{2}, & n_2 &= \frac{r_1 + r_2 - r_3 - r_4}{2}, \\ n_3 &= r_3 - \frac{r_1 + r_2}{2}, & n_4 &= \frac{r_1 - r_2}{2}. \end{aligned}$$

Moreover, we have

$$\frac{1}{2} n \tau_c {}^t n = \frac{\tau^*}{4} r_1^2 - \frac{1}{4\tau^*} r_2^2 + \frac{i}{4} r_3^2 + \frac{i}{4} r_4^2.$$

For $a, b \in \mathbb{Z}^4$, we set $\alpha = a {}^t L$ and $\beta = b L^{-1}$. The components of α and β are

$$\begin{aligned} (\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (a_1 + a_2 + a_3 + a_4, a_1 + a_2 + a_3 - a_4, a_1 + a_2 + 2a_3, a_1 - a_2), \\ (\beta_1, \beta_2, \beta_3, \beta_4) &= \left(\frac{b_1 + b_2 - b_3 + b_4}{2}, \frac{b_1 + b_2 - b_3 - b_4}{2}, b_3 - \frac{b_1 + b_2}{2}, \frac{b_1 - b_2}{2} \right). \end{aligned}$$

Consequently, we have

$$\begin{aligned} (4.8) \quad \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\tau_c) &= \frac{1}{4} \sum_{\nu_1, \nu_2 \in \{0,1\}} \mathbf{e} \left(-\frac{(\alpha_1 + \alpha_2)\nu_1 + (\alpha_3 + \alpha_4)\nu_2}{4} \right) \vartheta_{\alpha_1, \beta_1 + \nu_1} \left(\frac{\tau^*}{2} \right) \\ &\quad \times \vartheta_{\alpha_2, \beta_2 + \nu_1} \left(-\frac{1}{2\tau^*} \right) \vartheta_{\alpha_3, \beta_3 + \nu_2} \left(\frac{i}{2} \right) \vartheta_{\alpha_4, \beta_4 + \nu_2} \left(\frac{i}{2} \right). \end{aligned}$$

PROOF. Substituting the expressions for n_1, n_2, n_3, n_4 into the quadratic form, we obtain

$$\frac{1}{2i} n \tau_c {}^t n = -\frac{u+v}{4(u-v)} r_1^2 - \frac{u-v}{4(u+v)} r_2^2 + \frac{r_3^2}{4} + \frac{r_4^2}{4}.$$

The definition of τ^* gives the claimed quadratic identity. Every element of the image of $n \mapsto r$ satisfies $r_1 \equiv r_2$ and $r_3 \equiv r_4 \pmod{2}$, since $r_1 - r_2 = 2n_4$ and $r_3 - r_4 = 2(n_2 + n_3)$. Conversely, suppose that $r_1 \equiv r_2$ and $r_3 \equiv r_4 \pmod{2}$. Then the four displayed expressions for n_1, n_2, n_3, n_4 are integers, and substituting them back, we recover r , which proves that $n \mapsto r$ is a bijection. Since $r + \alpha/2 = (n + a/2) {}^t L$ and $\beta = b L^{-1}$, the defining series becomes

$$\begin{aligned} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\tau_c) &= \sum_{\substack{r \in \mathbb{Z}^4 \\ r_1 \equiv r_2 \pmod{2} \\ r_3 \equiv r_4 \pmod{2}}} \mathbf{e} \left(\frac{\tau^*}{4} \left(r_1 + \frac{\alpha_1}{2} \right)^2 - \frac{1}{4\tau^*} \left(r_2 + \frac{\alpha_2}{2} \right)^2 \right. \\ &\quad \left. + \frac{i}{4} \sum_{j=3}^4 \left(r_j + \frac{\alpha_j}{2} \right)^2 + \frac{1}{2} \sum_{j=1}^4 \left(r_j + \frac{\alpha_j}{2} \right) \beta_j \right). \end{aligned}$$

For integers r_1, r_2, r_3, r_4 , we have

$$\begin{aligned} \frac{1}{2} \sum_{\nu_1 \in \{0,1\}} \mathbf{e} \left(\frac{\nu_1(r_1 + r_2)}{2} \right) &= \frac{1}{2} (1 + (-1)^{r_1+r_2}), \\ \frac{1}{2} \sum_{\nu_2 \in \{0,1\}} \mathbf{e} \left(\frac{\nu_2(r_3 + r_4)}{2} \right) &= \frac{1}{2} (1 + (-1)^{r_3+r_4}). \end{aligned}$$

The sum over ν_1 is 1 if $r_1 \equiv r_2 \pmod{2}$ and 0 otherwise, and the sum over ν_2 has the same property for r_3 and r_4 . Multiplying the summand by these two sums, we replace the constrained sum by a sum over all $r \in \mathbb{Z}^4$ and obtain the factor $1/4$. For fixed $\nu_1, \nu_2 \in \{0,1\}$, the sum over $r \in \mathbb{Z}^4$ factors into four sums over r_1, r_2, r_3 , and r_4 . For example, the sum over r_1 is

$$\begin{aligned} \sum_{r_1 \in \mathbb{Z}} \mathbf{e} \left(\frac{\tau^*}{4} \left(r_1 + \frac{\alpha_1}{2} \right)^2 + \frac{\beta_1}{2} \left(r_1 + \frac{\alpha_1}{2} \right) + \frac{\nu_1 r_1}{2} \right) \\ = \mathbf{e} \left(-\frac{\alpha_1 \nu_1}{4} \right) \vartheta_{\alpha_1, \beta_1 + \nu_1} \left(\frac{\tau^*}{2} \right). \end{aligned}$$

Here $\nu_1 r_1/2 = \nu_1(r_1 + \alpha_1/2)/2 - \alpha_1 \nu_1/4$, which produces the factor $\mathbf{e}(-\alpha_1 \nu_1/4)$. The sums over r_2, r_3 , and r_4 give the remaining three theta constants in (4.8), together with the factors $\mathbf{e}(-\alpha_2 \nu_1/4)$, $\mathbf{e}(-\alpha_3 \nu_2/4)$, and $\mathbf{e}(-\alpha_4 \nu_2/4)$, respectively. The product of the four factors is

$$\mathbf{e} \left(-\frac{(\alpha_1 + \alpha_2)\nu_1 + (\alpha_3 + \alpha_4)\nu_2}{4} \right),$$

which gives (4.8). $\square$

Hereafter, we use the following classical identities for Jacobi's theta constants; cf. [BB98, Chapter 2]. We define the elliptic modular function λ by

$$\lambda(\tau) = \frac{\vartheta_{10}(\tau)^4}{\vartheta_{00}(\tau)^4}, \quad \tau \in \mathbb{H}.$$

The function λ restricts to a bijection from $i\mathbb{R}_{>0}$ onto $(0,1)$. For $\tau \in \mathbb{H}$, we have

$$(4.9) \quad \vartheta_{00}(\tau) = \vartheta_{00}(4\tau) + \vartheta_{10}(4\tau), \quad \vartheta_{01}(\tau) = \vartheta_{00}(4\tau) - \vartheta_{10}(4\tau),$$

$$(4.10) \quad \vartheta_{01}(\tau)^2 = \vartheta_{00}(2\tau)^2 - \vartheta_{10}(2\tau)^2, \quad \vartheta_{10}(\tau)^2 = 2\vartheta_{00}(2\tau)\vartheta_{10}(2\tau),$$

$$(4.11) \quad \vartheta_{00}(-1/\tau) = \sqrt{-i\tau} \vartheta_{00}(\tau), \quad \vartheta_{01}(-1/\tau) = \sqrt{-i\tau} \vartheta_{10}(\tau),$$

$$(4.12) \quad \vartheta_{00}(i) = \frac{\pi^{1/4}}{\Gamma(3/4)}, \quad \vartheta_{01}(i) = \vartheta_{10}(i) = 2^{-1/4} \vartheta_{00}(i),$$

$$(4.13) \quad \vartheta_{00}(\tau)^4 = \vartheta_{01}(\tau)^4 + \vartheta_{10}(\tau)^4, \quad \vartheta_{00}(\tau)^2 = F\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda(\tau)\right),$$

where $\sqrt{-i\tau}$ denotes the principal branch, which is real and positive for $\tau \in i\mathbb{R}_{>0}$. In the second identity of (4.13), the hypergeometric function is continued from its defining series at $\lambda(i) = 1/2$ along the image under λ of a path in $\mathbb{H}$ from i to τ . All the moduli to which (4.11) is applied below lie in $i\mathbb{R}_{>0}$. In (4.13), the two identities are due to Jacobi. We obtain (4.9) by separating the even and odd terms in the defining series; cf. [BB98, Section 2.1, equations (2.1.1)–(2.1.4)]. The first identities in (4.10) and (4.13) are equations (2.1.9) and (2.1.10), respectively, of [BB98, Section 2.1]. The second identity in (4.13) follows from [BB98, Theorem 2.1] by analytic continuation with the branch specified above. The transformation (4.11)

is the functional equation of Jacobi's theta constants; see [Mum07, Chapter I, Section 7]. The value $\vartheta_{00}(i)$ in (4.12) is given in [CM23]. By Proposition 4.3, we obtain the following corollary.

Corollary 4.4. *We have*

$$\begin{aligned}\Theta_1 &= \Theta_2 = \frac{\sqrt{2}}{4} \vartheta_{00}(i)^2 \left[\vartheta_{00}\left(\frac{\tau^*}{2}\right) \vartheta_{00}\left(-\frac{1}{2\tau^*}\right) + \vartheta_{01}\left(\frac{\tau^*}{2}\right) \vartheta_{01}\left(-\frac{1}{2\tau^*}\right) \right], \\ \Theta_3 &= -\Theta_4 = \frac{\sqrt{2}}{4} \vartheta_{00}(i)^2 \left[\vartheta_{01}\left(\frac{\tau^*}{2}\right) \vartheta_{00}\left(-\frac{1}{2\tau^*}\right) - \vartheta_{00}\left(\frac{\tau^*}{2}\right) \vartheta_{01}\left(-\frac{1}{2\tau^*}\right) \right].\end{aligned}$$

PROOF. Apply (4.8) to the four characteristics of (4.7). In the theta constants on the right-hand side of (4.8), we reduce the components of α and β modulo 2 using $\vartheta_{\alpha_j+2, \beta_j} = \vartheta_{\alpha_j, \beta_j}$ and $\vartheta_{\alpha_j, \beta_j+2} = \mathbf{e}(\alpha_j/2) \vartheta_{\alpha_j, \beta_j}$, which follow from (2.2) by a shift of the summation index. We keep the original components of α in the exponential factor and retain the factors produced by the second identity. For instance, $(a, b) = ((1, 1, 0, 0), 0)$ gives $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (2, 2, 2, 0)$ and $(\beta_1, \beta_2, \beta_3, \beta_4) = (0, 0, 0, 0)$, and we obtain

$$\Theta_1 = \frac{1}{4} \left[\vartheta_{00}\left(\frac{\tau^*}{2}\right) \vartheta_{00}\left(-\frac{1}{2\tau^*}\right) + \vartheta_{01}\left(\frac{\tau^*}{2}\right) \vartheta_{01}\left(-\frac{1}{2\tau^*}\right) \right] \left[\vartheta_{00}\left(\frac{i}{2}\right)^2 - \vartheta_{01}\left(\frac{i}{2}\right)^2 \right].$$

Applying (4.11) at $\tau = 2i$, we have $\vartheta_{00}(\frac{i}{2}) = \sqrt{2} \vartheta_{00}(2i)$ and $\vartheta_{01}(\frac{i}{2}) = \sqrt{2} \vartheta_{10}(2i)$. Using these identities with (4.10) at $\tau = i$ and (4.12), we find that the second factor equals $2\vartheta_{01}(i)^2 = \sqrt{2} \vartheta_{00}(i)^2$. The following table lists the characteristic (a, b) and the vectors $(\alpha_1, \dots, \alpha_4)$ and $(\beta_1, \dots, \beta_4)$ for each of $\Theta_1, \dots, \Theta_4$.

	(a, b)	$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$	$(\beta_1, \beta_2, \beta_3, \beta_4)$
Θ_1	$((1, 1, 0, 0), 0)$	$(2, 2, 2, 0)$	$(0, 0, 0, 0)$
Θ_2	$(0, (1, 1, 0, 0))$	$(0, 0, 0, 0)$	$(1, 1, -1, 0)$
Θ_3	$((0, 0, 1, 1), (1, 1, 1, 1))$	$(2, 0, 2, 0)$	$(1, 0, 0, 0)$
Θ_4	$((1, 1, 1, 1), (0, 0, 1, 1))$	$(4, 2, 4, 0)$	$(0, -1, 1, 0)$

For Θ_2 , the first factor is the same as for Θ_1 . The second factor is $2\vartheta_{00}(\frac{i}{2})\vartheta_{01}(\frac{i}{2}) = 4\vartheta_{00}(2i)\vartheta_{10}(2i) = 2\vartheta_{10}(i)^2 = \sqrt{2} \vartheta_{00}(i)^2$. Hence we have $\Theta_2 = \Theta_1$. For Θ_3 , the exponential factor in (4.8) equals $(-1)^{\nu_1}(-1)^{\nu_2}$. The sum over ν_1 gives the difference $\vartheta_{01}(\frac{\tau^*}{2})\vartheta_{00}(-\frac{1}{2\tau^*}) - \vartheta_{00}(\frac{\tau^*}{2})\vartheta_{01}(-\frac{1}{2\tau^*})$ in the stated formula for Θ_3 . The sum over ν_2 gives $\vartheta_{00}(\frac{i}{2})^2 - \vartheta_{01}(\frac{i}{2})^2 = \sqrt{2} \vartheta_{00}(i)^2$, and the two sums together yield the stated formula for Θ_3 . For Θ_4 , the exponential factor equals $(-1)^{\nu_1}$. Note that $\alpha_3 + \alpha_4 = 4$. Using $\vartheta_{0,2}(\frac{i}{2}) = \vartheta_{00}(\frac{i}{2})$, we have

$$\Theta_4 = \frac{1}{4} \left[\vartheta_{00}\left(\frac{\tau^*}{2}\right) \vartheta_{01}\left(-\frac{1}{2\tau^*}\right) - \vartheta_{01}\left(\frac{\tau^*}{2}\right) \vartheta_{00}\left(-\frac{1}{2\tau^*}\right) \right] \cdot 2\vartheta_{00}\left(\frac{i}{2}\right) \vartheta_{01}\left(\frac{i}{2}\right) = -\Theta_3. \quad \square$$

The following is the key lemma of this section.

Lemma 4.5. *We have*

$$\Theta_1 \Theta_3 = \frac{-i\tau^*}{4} \vartheta_{00}(i)^4 \left(\vartheta_{01}(\tau^*)^4 - \vartheta_{10}(\tau^*)^4 \right).$$

PROOF. By (4.11) at $\tau = 2\tau^*$, where $\sqrt{-2i\tau^*}$ is positive, we have

$$\vartheta_{00}\left(-\frac{1}{2\tau^*}\right) = \sqrt{-2i\tau^*} \vartheta_{00}(2\tau^*), \quad \vartheta_{01}\left(-\frac{1}{2\tau^*}\right) = \sqrt{-2i\tau^*} \vartheta_{10}(2\tau^*).$$

Using (4.9) at $\tau^*/2$ and (4.10) at τ^* , we obtain

$$\vartheta_{00}\left(\frac{\tau^*}{2}\right) \vartheta_{00}(2\tau^*) + \vartheta_{01}\left(\frac{\tau^*}{2}\right) \vartheta_{10}(2\tau^*) = \vartheta_{01}(\tau^*)^2 + \vartheta_{10}(\tau^*)^2,$$

$$\vartheta_{01}\left(\frac{\tau^*}{2}\right)\vartheta_{00}(2\tau^*) - \vartheta_{00}\left(\frac{\tau^*}{2}\right)\vartheta_{10}(2\tau^*) = \vartheta_{01}(\tau^*)^2 - \vartheta_{10}(\tau^*)^2.$$

Substituting these four identities into the formulas of Corollary 4.4, we obtain

$$\begin{aligned}\Theta_1 &= \frac{1}{2}\vartheta_{00}(i)^2\sqrt{-i\tau^*}(\vartheta_{01}(\tau^*)^2 + \vartheta_{10}(\tau^*)^2), \\ \Theta_3 &= \frac{1}{2}\vartheta_{00}(i)^2\sqrt{-i\tau^*}(\vartheta_{01}(\tau^*)^2 - \vartheta_{10}(\tau^*)^2).\end{aligned}$$

Multiplying these two equalities, we obtain the stated formula. $\square$

5. EVALUATION OF THE CONSTANT

In this section, we identify the modulus τ^* with a value of a Schwarz map of the Gauss hypergeometric differential equation, and complete the evaluation of the constant κ .

A Schwarz map is a ratio of two linearly independent solutions of the Gauss hypergeometric differential equation with parameters $\frac{1}{2}, \frac{1}{2}; 1$, and is determined only up to a Möbius transformation. We use the ratio exhibited in (5.1). We begin with the following lemma.

Lemma 5.1. *We have*

$$(5.1) \quad \tau^* = i \frac{\mathcal{F}(1-m)}{\mathcal{F}(m)},$$

and consequently

$$(5.2) \quad \sqrt{c}\vartheta_{00}(\tau^*)^4 = \vartheta_{01}(\tau^*)^4 - \vartheta_{10}(\tau^*)^4,$$

$$(5.3) \quad \mathcal{K}_2 = \frac{\sqrt{2}\pi\vartheta_{00}(\tau^*)^2}{\sqrt{c}},$$

$$(5.4) \quad {}^t\eta_c U\eta_c = u^2 - v^2 = -8i\tau^*\mathcal{K}_2^2 = -\frac{16\pi^2 i\tau^*\vartheta_{00}(\tau^*)^4}{c}.$$

PROOF. By (4.1) and Lemma 4.2, we have

$$\tau^* = i \left(\frac{\sqrt{2}\mathcal{K}_1}{\mathcal{K}_2} - 1 \right) = i \frac{\mathcal{F}(1-m)}{\mathcal{F}(m)}, \quad u^2 - v^2 = -8i\tau^*\mathcal{K}_2^2 > 0.$$

This proves (5.1); the positivity of $u^2 - v^2$ is established in Lemma 4.2. By [BB98, Theorem 2.3(b)], with $k = \sqrt{m}$, we have $\lambda(i\mathcal{F}(1-m)/\mathcal{F}(m)) = m$. By (5.1), we have $\tau^* = i\mathcal{F}(1-m)/\mathcal{F}(m)$, and hence $\lambda(\tau^*) = m$. Since $\tau^* \in i\mathbb{R}_{>0}$ and λ restricts to a bijection from $i\mathbb{R}_{>0}$ onto $(0, 1)$, the modulus τ^* is the unique point of $i\mathbb{R}_{>0}$ at which λ takes the value m . The second identity in (4.13) then gives $\vartheta_{00}(\tau^*)^2 = \mathcal{F}(\lambda(\tau^*)) = \mathcal{F}(m)$. Equation (5.2) follows from $1 - 2\lambda(\tau^*) = (\vartheta_{01}(\tau^*)^4 - \vartheta_{10}(\tau^*)^4)/\vartheta_{00}(\tau^*)^4 = \sqrt{c}$, where we use the first identity in (4.13). Equation (5.3) follows from $\mathcal{K}_2 = \sqrt{2}\pi c^{-1/2}\mathcal{F}(m) = \sqrt{2}\pi c^{-1/2}\vartheta_{00}(\tau^*)^2$. Finally, we substitute (5.3) into the identity $u^2 - v^2 = -8i\tau^*\mathcal{K}_2^2$ established at the beginning of the proof, and obtain the last expression in (5.4). $\square$

PROOF OF THE MAIN THEOREM. By (4.7), Corollary 4.4, Lemma 4.5, and (5.2) of Lemma 5.1, we have

$$f_{J^*}(\eta_c) = \frac{\vartheta_{00}(i)^8}{4} (2\Theta_1^2)(2\Theta_3^2) = \vartheta_{00}(i)^8 (\Theta_1\Theta_3)^2$$

$$= \vartheta_{00}(i)^8 \left(\frac{-i\tau^*}{4} \vartheta_{00}(i)^4 \sqrt{c} \vartheta_{00}(\tau^*)^4 \right)^2 = -\frac{\tau^{*2}}{16} \vartheta_{00}(i)^{16} c \vartheta_{00}(\tau^*)^8.$$

On the other hand, by (5.4) of Lemma 5.1, we have

$$({}^t\eta_c U \eta_c)^2 = -\frac{256\pi^4 \tau^{*2} \vartheta_{00}(\tau^*)^8}{c^2}.$$

Substituting these two expressions into (4.5), we obtain

$$\kappa = \frac{-\frac{\tau^{*2}}{16} \vartheta_{00}(i)^{16} c \vartheta_{00}(\tau^*)^8}{\left(-\frac{256\pi^4 \tau^{*2} \vartheta_{00}(\tau^*)^8}{c^2} \right) (-c^3)} = -\frac{\vartheta_{00}(i)^{16}}{4096 \pi^4} = -\frac{\vartheta_{00}(i)^{16}}{(8\pi)^4} = -\frac{1}{2^{12} \Gamma(3/4)^{16}},$$

where the last equality uses $\vartheta_{00}(i) = \pi^{1/4}/\Gamma(3/4)$. This computation determines the value κ of the function G , and the Main Theorem follows for every $(2, 2, 2, 2)$ -partition J . $\square$

Combining the Main Theorem with the continuous extensions of f_{J^*} and of the theta constants to the degenerate configuration, we obtain the boundary value

$$f_{J^*}(\eta_c) = \frac{\vartheta_{00}(i)^{16}}{(8\pi)^4} c^3 ({}^t\eta_c U \eta_c)^2 = \frac{c^3}{2^{12} \Gamma(3/4)^{16}} ({}^t\eta_c U \eta_c)^2.$$

Remark 5.2. The four branch points $0, c, 1, \infty$ of the limit curve $C_c: w^4 = z^3(z - c)^3(z - 1)$ carry the weights $\frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}$ of $\frac{dz}{w}$, respectively. The sum of these weights is 2. The curves C_c with $c \in (0, 1)$ thus form one of the one-dimensional families of [DM86]. The modulus τ^* satisfies $\lambda(\tau^*) = (1 - \sqrt{c})/2$. The involution $\rho^2: C_c \ni (z, w) \mapsto (z, -w) \in C_c$ lifts to the normalization, and the substitution $y = w^2/(z(z - c))$ identifies the quotient of the normalization by ρ^2 with the Legendre elliptic curve $y^2 = z(z - c)(z - 1)$. The same degeneration performed only at $x_4 = x_5 = x_6 = c$ leads to the three-dimensional complex ball attached to six points. The Thomae-type formulas for six points are the counterparts of the Main Theorem, with the constant $1/((16\pi)^2 \Gamma(3/4)^8)$; see [MN26].

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