

TRANSFORMATIONS OF LAURICELLA'S F_D AND CORRESPONDENCES BETWEEN HYPERELLIPTIC CURVES

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ABSTRACT. We construct rational coordinates on the curves defined by norm equations for quintic and sextic polynomial families and parametrize the real paths on which all zeros of the selected factors lie in the left half-plane. We use these coordinates to obtain Lauricella integral representations with specified branches for every nonnegative parameter of both pencils. We construct finite correspondences from smooth curves of genera seven and four to smooth curves of genus two at exponent one half. We show that the adjoint homomorphisms of Jacobians are twice homomorphisms whose adjoint compositions equal multiplication by four and three, respectively, and that these latter homomorphisms are not divisible by an integer greater than one. We prove that these latter homomorphisms embed the source Jacobians, with induced polarization types $(4, 4)$ and $(3, 3)$, respectively.

1. INTRODUCTION

Lauricella's function F_D is defined by the series (2.1). We study its restriction

$$F_D^{(N)}\left(\beta + \frac{1}{2}; \beta^{[N]}; N\beta; x_1, \dots, x_N\right), \quad N \in \{5, 6\},$$

where β is a complex parameter and $\beta^{[N]}$ denotes N copies of β . The arguments are determined by a polynomial pencil and depend on the real family parameter $\sigma \geq 1$ and the real pencil parameter E . The Euler integral for F_D is given in [MO09, Eq. (1)].

In this paper, we use a norm identity and a trace formula for odd power sums of the roots to construct algebraic correspondences. For the specified quintic and sextic families, we use rational coordinates on conics to parametrize the whole real paths. We summarize the geometric result for the two pencils of Theorems 4.1 and 5.1. For the quintic pencil, we use Q_0 from (4.1) and M, α from (4.4). For the sextic pencil, we put $Q_0 = H^2$, where H , α , and M are defined in (5.1), (5.4), and (5.5), respectively. For $E \in \mathbb{C}$, we put

$$\begin{aligned} C_E : \quad & y^2 = Q_0(v) + Ev, \\ T_E : \quad & \xi^2 = M(r) - Er^8/\alpha^2 \quad \text{for the quintic pencil,} \\ T_E : \quad & \xi^2 = M(x) - Ex^5/\alpha^2 \quad \text{for the sextic pencil.} \end{aligned}$$

The same letters denote the smooth projective models when the defining polynomials are squarefree. For each fixed σ , the defining polynomials of C_E and T_E are squarefree for all but finitely many values of E . Then we have the following theorem.

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Theorem 1.1. *Suppose that $\sigma \geq 1$ and $E \in \mathbb{C}$, with Q_0, M, α as in (4.1) and (4.4) for the quintic pencil, or (5.1), (5.4), and (5.5) for the sextic pencil. Suppose also that the defining polynomials of C_E and T_E are squarefree. There is a finite correspondence*

$$C_E \xleftarrow{p} D_E \xrightarrow{q} T_E$$

whose degrees and genera, together with the integer in the composition formula below, are

	$g(C_E)$	$g(T_E)$	$\deg p$	$\deg q$	m
quintic	2	7	16	5	16
sextic	2	4	10	6	12.

Here D_E is the disjoint union of the compact normalizations of all components, and the degrees include all components. The homomorphism

$$\Psi = p_* q^* : \text{Jac}(T_E) \longrightarrow \text{Jac}(C_E)$$

is surjective. For the adjoint $\Psi^\dagger = q_* p^*$ with respect to the canonical principal polarizations, the composition is multiplication by m on $\text{Jac}(C_E)$:

$$\Psi \Psi^\dagger = [m].$$

We put $f = \Psi^\dagger$ and $n = m/4$. There is a unique homomorphism $f_0 : \text{Jac}(C_E) \rightarrow \text{Jac}(T_E)$ such that

$$f = 2f_0, \quad f_0^\dagger f_0 = [n], \quad f_0^*[\Theta_T] = n[\Theta_C].$$

Here $[\Theta_C], [\Theta_T]$ are the canonical principal polarization classes. The homomorphism f_0 is not an integer multiple of another homomorphism with $d \geq 2$. The homomorphism f_0 is an embedding, and the induced polarization on its image has type (n, n) , namely $(4, 4)$ for the quintic pencil and $(3, 3)$ for the sextic pencil. The kernel of f is exactly $\text{Jac}(C_E)[2]$. The adjoint $f_0^\dagger$ is surjective with connected kernel. By addition, we obtain an isogeny from the product of $\text{im } f_0$ and $\ker f_0^\dagger$ to $\text{Jac}(T_E)$. This isogeny has degree n^4 , namely 256 in the quintic case and 81 in the sextic case.

Theorem 6.11 gives the correspondence and the traces of a basis of regular differentials. Theorem 6.12 computes the composition and its polarization consequences. Proposition 6.14 proves the divisibility by two in the group of homomorphisms. Theorem 6.17 determines the kernel, and Corollary 6.18 gives the induced polarizations and isogeny degrees. In particular, the correspondence specifies a rational idempotent of the target Jacobian. These assertions concern smooth curves at $\beta = 1/2$. We treat the general complex exponent by Mellin integrals and analytic continuation.

The quintic correspondence is obtained from the construction of Kanev and Lange [KL08, §§2.13–2.14 and Remark 3.9, author manuscript arXiv:0707.0364v1]. Proposition 6.19 identifies the selected component and gives the ramification of the double cover in the construction. Pullback and norm maps are adjoint for the canonical principal polarizations; see [KM16, § 2, (2.1)–(2.3), (4.6)]. The cotangent action of a correspondence is computed by pullback followed by a field trace [KM16, (4.1)–(4.2)]. Theta-divisor embeddings are related to hyperelliptic curves of genus four whose Jacobians contain complementary surfaces with restricted polarization of type $(1, 3)$ in [BS17, Theorem 3.8]. Polarized $(2, 2)$ decompositions are related

to double covers and Prym varieties in [BK25, Theorem 1.1]. Isogeny decompositions are described by rational idempotents in [BK25, Remark 1.2]. Lange and Ortega show that a connected non-cyclic étale triple covering of a curve of genus two has a hyperelliptic covering curve of genus four and that its Prym variety, with its principal polarization, is a Jacobian of genus two; see [LO11, Corollary 3.4 and Theorems 4.9 and 4.17, arXiv:0911.5224v2]. The canonical principal polarization of the Jacobian of the covering curve induces a polarization of type $(3, 3)$ on the Prym variety; see [LO11, Proposition 3.3, the same author manuscript]. For the specified sextic family, we construct the curves and the correspondence explicitly and determine $\ker \Psi^\dagger$ and $\ker f_0$, where $\Psi^\dagger = 2f_0$, whenever the defining polynomials of both curves are squarefree. The real path of spectral factors also gives the transformation in Theorem 5.1 with its specified branches. We compute the pullback of the canonical principal polarization of $\text{Jac}(T_E)$ by $\Psi^\dagger$, which has finite kernel. We prove that $\ker \Psi^\dagger$ is exactly the two-torsion subgroup of $\text{Jac}(C_E)$. After division by two in the group of homomorphisms of abelian varieties, monodromy excludes a nonzero kernel in every smooth fiber. For the quintic pencil, continuation of the roots around loops induces an irreducible action on the two-torsion subgroup. For the sextic pencil, the monodromy at infinity acts irreducibly on the three-torsion subgroup. Thus the correspondences identify the source Jacobians with polarized subvarieties of the target Jacobians; see Corollary 6.18.

For $P(z) = \prod_{j=1}^N (1 + p_j z)$, we consider the coefficient equation

$$P(\sqrt{-1}\sqrt{v})P(-\sqrt{-1}\sqrt{v}) = Q_0(v) + \mu v,$$

where $Q_0 \in \mathbb{R}[v]$, $\deg Q_0 = N$, $Q_0(0) = 1$, $Q_0(v) > 0$ for $v \geq 0$, and $\mu \geq 0$. We choose the numbers p_j with $\text{Re } p_j > 0$ and put $\mathcal{A}_-(\mu) = \sum_j p_j^{-1}$. The coefficients of P define an algebraic curve, and the inequalities $\text{Re } p_j > 0$ select a real path on that curve. The resulting identity is $P(z)P(-z) = Q_0(-z^2) - \mu z^2$, with $P(0) = 1$ and all zeros of P in the left half-plane. Selecting the factor with these zeros is polynomial spectral factorization. We use this factorization to construct rational coordinates on the selected components of the curve defined by the norm equation and to determine their whole real paths. The norm identity alone gives the algebraic correspondence and expresses the traces through odd power sums of the roots; see Proposition 6.5. For the specified families, the coefficient equations reduce to double covers with discriminants of degree six. We remove the squared factors to obtain conics and use their rational coordinates to express $\mathcal{A}_-$ as rational functions whose derivatives consist of two monomials. We choose the two families by their factorizable fibers in (4.1)–(4.2) and (5.2). These fibers give squared factors in the discriminants obtained by eliminating coefficients; after removing these factors, we obtain conics with rational coordinates. We then verify that these coordinates cover the real paths; factorization at the selected fibers alone does not prove this assertion. The positive integral for $\mathcal{A}'_-$ identifies the whole real paths and proves that the pencil parameter has nonzero derivative at their finite endpoints. Both real paths include $E = 0$. For $\sigma > 1$, the factorizable sextic fiber in (5.2) is an explicit interior point of this path. The continuation in Lemma 3.3 includes positive integer values of β .

The number of Lauricella terms records the number of monomials in the polynomial factor of the transformed integrand. The finite expansion with upper parameters -1 also gives a single $F_D^{(17)}$ or $F_D^{(10)}$; see Fact 2.2 and (5.10)–(5.11). We

use the two-term forms to retain β as the repeated upper parameter of each Lauricella function. Transformations by polynomial substitutions and by two orders of integration are treated in [EK24, §2] and [WYZ26, §§5.2.1–5.2.2], respectively. For integer samples of powers of a reciprocal quadratic rational function, the generating function is reduced to a Mellin integral with a quadratic denominator by Raghava [Rag26, Eqs. (1.4), (6.2)–(6.4)]. Here we use Mellin integrals and rational coordinates on the real paths of spectral factors to construct Lauricella transformations; see Proposition 3.2 and Theorems 4.1 and 5.1. The real paths also identify two real Abelian integrals, with specified endpoints, as integrals of the explicit traces on T_E .

At $\sigma = 1$, Corollaries 4.2 and 5.2 express the source functions by ${}_5F_4$. The sums of the reciprocals of the numbers p_j at this parameter belong to the algebraic hypergeometric family in [Mai14, Theorem 4.4(ii), arXiv:0906.3485v4]. To specify the parameters and branches, we take $N = 5$ or 6 , $0 < \mu \ll 1$, and put $t_j = 1 - p_j^{-2}$ and $\hat{t}_j = \mu^{-1/N} t_j$, with the positive real N th root of μ . Then

$$t_j^N + \mu t_j - \mu = 0, \quad \hat{t}_j^N + \mu^{1/N} \hat{t}_j - 1 = 0.$$

We put $\varepsilon_N = \exp(2\pi i/N)$ and number the roots so that $\hat{t}_j \rightarrow \varepsilon_N^{-(j-1)}$ as $\mu \rightarrow 0^+$. In Maier's notation, we take $p = 1$, $q = N - 1$, $n = N$, $g = -\mu^{1/N}$, $\beta = 1$, $a = -1/2$, and $\ell = \kappa = 0$. His variable then satisfies $\zeta^{-1} = -(N - 1)^{N-1} \mu/N^N$, and the powers of the roots in his sum satisfy

$$[\varepsilon_N^{j-1} \hat{t}_j]^{N/2} = (1 - t_j)^{1/2} = p_j^{-1}.$$

All powers in this equality are the branches tending to 1 as $\mu \rightarrow 0^+$. Since his function F_0 equals 1, his theorem represents $N^{-1} \sum_j p_j^{-1}$ near $\mu = 0$. The choice $\operatorname{Re} p_j > 0$ continues this sum along $\mu > 0$; at multiple roots, the sum is interpreted by continuity as in Section 3. For $\sigma > 1$, the quintic and sextic pencils have fibers with multiplicity partitions $2 + 2 + 1$ and $2 + 2 + 2$, respectively. These partitions do not occur in the pencils $(1 + v)^N + \mu v$. The multiplicity partitions exclude equivalence under a Möbius change of v and a projective change of the pencil generators; see Proposition 6.3.

The construction also has a relation to the coefficient method in *Integer Parameter Shifts in One-Variable Restrictions of Lauricella's F_C* [Nak26]. In both papers, we start from a product of factors at cyclically related arguments and obtain Lauricella representations from a rationality property. Here we construct rational coordinates on a curve defined by the norm equation and use Mellin integrals to obtain the Lauricella representations. In the F_C paper, the shifted coefficients are products of rational functions of the index and the terms of one coefficient sequence, and the passage uses multiplication by upper Pochhammer factors. The two methods use the rationality of different intermediate objects under different hypotheses.

2. SERIES AND EULER INTEGRALS

In this section, we collect the series and integral formulas used in the proofs. For $z \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$, we put $(z)_0 = 1$ and $(z)_m = z(z+1) \cdots (z+m-1)$. We fix a positive integer k . For $\mathbf{m} = (m_1, \dots, m_k) \in \mathbb{Z}_{\geq 0}^k$ and $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{C}^k$, we put

$$|\mathbf{m}| = \sum_{i=1}^k m_i, \quad \mathbf{m}! = \prod_{i=1}^k m_i!, \quad \mathbf{x}^{\mathbf{m}} = \prod_{i=1}^k x_i^{m_i}.$$

We use the definition

$$(2.1) \quad F_D^{(k)}(a; b_1, \dots, b_k; c; \mathbf{x}) = \sum_{\mathbf{m} \geq 0} \frac{(a)_{|\mathbf{m}|} \prod_i (b_i)_{m_i}}{(c)_{|\mathbf{m}|} \mathbf{m}!} \mathbf{x}^{\mathbf{m}}.$$

Here the lower parameter avoids $\mathbb{Z}_{\leq 0}$. The series converges absolutely on $\max_i |x_i| < 1$. We put

$${}_pF_q \left(\begin{matrix} u_1, \dots, u_p \\ v_1, \dots, v_q \end{matrix}; z \right) = \sum_{m \geq 0} \frac{\prod_{j=1}^p (u_j)_m}{\prod_{j=1}^q (v_j)_m} \frac{z^m}{m!}.$$

We use ${}_pF_q$ with $p = q + 1$ as its ordinary series for $|z| < 1$ and then continue the germ along the specified path.

We put $B(u, v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$.

We specify the branches in the Euler formula before stating the formula. We take $x_j \in \mathbb{C} \setminus [1, \infty)$ and use the logarithms of $1 - x_j t$ that vanish at $t = 0$. These logarithms exist on $0 \leq t \leq 1$. Then we have the following fact.

Fact 2.1. *For $\operatorname{Re} c > \operatorname{Re} a > 0$, we have*

$$(2.2) \quad \begin{aligned} & F_D^{(k)}(a; b_1, \dots, b_k; c; x_1, \dots, x_k) \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} \prod_{j=1}^k (1-x_j t)^{-b_j} dt; \end{aligned}$$

see [MO09, Eq. (1)].

For $P(z) = \prod_{j=1}^N (1 + p_j z)$, we define the norm for the quadratic extension given by $z^2 = -v$ by

$$\mathcal{N}_2[P](v) = P(\sqrt{-1}\sqrt{v})P(-\sqrt{-1}\sqrt{v}) = \prod_{j=1}^N (1 + p_j^2 v).$$

We fix polynomials Q, R with $Q(0) = 1$, $R(0) = 0$, and $\deg R < \deg Q = N$. We also fix the constant coefficient of P and one value of its leading coefficient. Then the equation

$$(2.3) \quad \mathcal{N}_2[P](v) = Q(v) + \lambda R(v)$$

defines a curve in the coefficients of P . The parameter λ determines the polynomial in the pencil. We write $\beta^{[m]}$ for a list of m copies of the complex parameter β . When a polynomial is written as $\prod_j (1 - x_j t)$, we count the numbers x_j with multiplicity and append zeros if the degree drops.

We record the finite expansion obtained by multiplying the Euler integrand by a polynomial.

Fact 2.2 ([EK24, Theorems 6–7, Eqs. (13), (15)]). *For positive integers n, m and complex parameters $a, b_1, \dots, b_n, c$ with $c \notin \mathbb{Z}_{\leq 0}$, we have the following identity of germs at the origin.*

$$F_D^{(n+m)} \left(a; b_1, \dots, b_n, \underbrace{-1, \dots, -1}_m; c; z_1, \dots, z_n, w_1, \dots, w_m \right)$$

$$= \sum_{S \subseteq \{1, \dots, m\}} (-1)^{|S|} \left(\prod_{j \in S} w_j \right) \frac{(a)_{|S|}}{(c)_{|S|}} F_D^{(n)}(a + |S|; b_1, \dots, b_n; c + |S|; z_1, \dots, z_n).$$

The identity continues when both sides follow the same paths in their arguments and the Euler integrals use the same logarithms.

PROOF. For $\operatorname{Re} c > \operatorname{Re} a > 0$, expand $\prod_{j=1}^m (1 - w_j t)$ in the Euler integral (2.2). The beta factor for a term of degree $|S|$ changes by $(a)_{|S|}/(c)_{|S|}$. Coefficient comparison and meromorphic continuation give the stated identity near the origin and its continuation along the same paths on both sides. $\square$

3. MELLIN INTEGRALS AND REAL PATHS

In this section, we express Mellin integrals by sums of the numbers p_j and their reciprocals. Throughout this section, we assume

$$(3.1) \quad \begin{aligned} Q, R \in \mathbb{R}[v], \quad Q(0) = 1, \quad Q(v) > 0 \ (v \geq 0), \quad R(v) > 0 \ (v > 0), \\ R(0) = 0, \quad M = \deg R < N = \deg Q, \quad m = \operatorname{ord}_{v=0} R \geq 1. \end{aligned}$$

For $\lambda \geq 0$, we set $Q_\lambda = Q + \lambda R$ and denote the roots of Q_λ by $\rho_1(\lambda), \dots, \rho_N(\lambda)$. We choose

$$p_j(\lambda) = (-\rho_j(\lambda))^{-1/2}, \quad \operatorname{Re} p_j(\lambda) > 0, \quad P_\lambda(z) = \prod_{j=1}^N (1 + p_j(\lambda)z).$$

We have $P_\lambda(z)P_\lambda(-z) = Q_\lambda(-z^2)$ and $P_\lambda(0) = 1$. Every zero of P_λ lies in the left half-plane. The factor determined by these conditions is the spectral factor of $Q_\lambda(-z^2)$. The coefficients of P_λ , with $\lambda \geq 0$, define a real path on the curve determined by (2.3). The polynomial P_λ has real coefficients and satisfies (2.3). Its leading coefficient is the positive square root of the leading coefficient of Q . The coefficients of P_λ are real analytic in λ , including values at which roots of Q_λ coincide. For a fixed $\lambda_0 \geq 0$, we choose positively oriented contours γ in $\mathbb{C} \setminus [0, \infty)$ enclosing all roots of Q_{λ_0} . The same contours work for all polynomials with coefficients near those of Q_{λ_0} , and

$$\sum_{j=1}^N p_j(\lambda)^k = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} (-z)^{-k/2} \frac{\partial_z Q_\lambda(z)}{Q_\lambda(z)} dz, \quad k = -1, 1, \dots, N.$$

These contour integrals are holomorphic in the coefficients of Q_λ . Newton's identities give the asserted analyticity of the coefficients of P_λ .

We put

$$\mathcal{A}_+(\lambda) = \sum_{j=1}^N p_j(\lambda), \quad \mathcal{A}_-(\lambda) = \sum_{j=1}^N p_j(\lambda)^{-1}.$$

We use these symmetric expressions in the Mellin integral identities. The following lemma gives the two kernels used below.

Lemma 3.1. *Under (3.1), we have*

$$(3.2) \quad 2\pi \mathcal{A}'_+(\lambda) = \int_0^\infty v^{-3/2} \frac{R(v)}{Q_\lambda(v)} dv,$$

$$(3.3) \quad 2\pi\mathcal{A}'_-(\lambda) = \int_0^\infty v^{-1/2} \frac{R(v)}{Q_\lambda(v)} dv.$$

Both derivatives are positive for $\lambda \geq 0$.

PROOF. We assume first that the roots of Q_λ are simple. Differentiating $Q_\lambda(\rho_j) = 0$ gives

$$\rho'_j = -\frac{R(\rho_j)}{\partial_v Q_\lambda(\rho_j)}, \quad \mathcal{A}'_+ = \frac{1}{2} \sum_j \frac{(-\rho_j)^{-1/2} R(\rho_j)}{\rho_j \partial_v Q_\lambda(\rho_j)}.$$

By the assumptions on the degrees and the constant terms, we have

$$\frac{R(v)}{vQ_\lambda(v)} = \sum_j \frac{R(\rho_j)}{\rho_j \partial_v Q_\lambda(\rho_j)} \frac{1}{v - \rho_j}.$$

We use the beta integral [DLMF, (5.12.3)], also given in [OLBC10, Chapter 5], in the form

$$\int_0^\infty \frac{v^{-1/2}}{v - \rho} dv = \pi(-\rho)^{-1/2}, \quad \rho \in \mathbb{C} \setminus [0, \infty).$$

Substituting the beta integral into the partial fraction expansion, we obtain (3.2). To include multiple roots, we fix $\lambda_0 \geq 0$ and replace Q by $Q_\varepsilon(v) = Q(v) + \varepsilon v$, where $\varepsilon > 0$. This replacement preserves (3.1) and the leading coefficient, since $N \geq 2$. All but finitely many ε give simple roots at λ_0 . To see this, we put $f(v) = Q(v) + \lambda_0 R(v)$. If $f(v) + \varepsilon v$ has a multiple root ρ , then $\rho \neq 0$ and $\rho f'(\rho) - f(\rho) = 0$. This last polynomial has constant term -1 , so it has only finitely many roots, each determining one value of ε . The contour integral gives the convergence of $\mathcal{A}'_+(\lambda_0)$ as $\varepsilon \downarrow 0$. On the right-hand side of (3.2), the perturbed integrand is bounded by the unperturbed one, which has orders $v^{m-3/2}$ at zero and $v^{M-N-3/2}$ at infinity. Dominated convergence proves (3.2) at λ_0 . This argument also covers pencils in which Q_λ has a multiple root for every value of λ .

Let $q_N > 0$ be the leading coefficient of Q . We put

$$\widehat{Q}(v) = q_N^{-1} v^N Q(1/v), \quad \widehat{R}(v) = q_N^{-1} v^N R(1/v).$$

These polynomials satisfy (3.1). The numbers obtained by the same choice of square roots for the pencil $\widehat{Q} + \lambda \widehat{R}$ are p_j^{-1} . We apply (3.2) to this pencil and substitute $v = 1/u$, which gives (3.3). Both kernels are positive. $\square$

By this lemma, we obtain the following proposition.

Proposition 3.2. *Under (3.1), we have*

$$(3.4) \quad \int_0^\infty v^{-3/2} \left(\frac{R(v)}{Q(v)} \right)^\beta dv = 2 \sin(\pi\beta) \int_0^\infty \lambda^{-\beta} \mathcal{A}'_+(\lambda) d\lambda,$$

$$(3.5) \quad \int_0^\infty v^{-1/2} \left(\frac{R(v)}{Q(v)} \right)^\beta dv = 2 \sin(\pi\beta) \int_0^\infty \lambda^{-\beta} \mathcal{A}'_-(\lambda) d\lambda,$$

where $1/(2m) < \operatorname{Re} \beta < 1$ in (3.4) and $1/(2(N-M)) < \operatorname{Re} \beta < 1$ in (3.5). All integrals converge absolutely, and powers of positive real numbers are defined by the real logarithm.

PROOF. We use the beta integral [DLMF, (5.12.3)] and the reflection formula [DLMF, (5.5.3)] in the form

$$\int_0^\infty \lambda^{-\beta} \frac{R(v)}{Q(v) + \lambda R(v)} d\lambda = \frac{\pi}{\sin(\pi\beta)} \left(\frac{R(v)}{Q(v)} \right)^\beta, \quad 0 < \operatorname{Re} \beta < 1.$$

For real β , Tonelli's theorem applies to each nonnegative kernel. The integrand of the source integral in (3.4) has orders $v^{m\beta-3/2}$ at zero and $v^{-(N-M)\beta-3/2}$ at infinity. For the source integral in (3.5), the orders are $v^{m\beta-1/2}$ at zero and $v^{-(N-M)\beta-1/2}$ at infinity. The stated bounds on β give the convergence of the source integrals. For complex β , we apply the same estimates with $\operatorname{Re} \beta$ in place of β and obtain the absolute convergence of the source integrals. Fubini's theorem and Lemma 3.1 give both equalities. $\square$

The following lemma specifies the continuation of the target Euler integrals in the parameter β .

Lemma 3.3. *Let $0 \leq \beta_0 < 1$ and $\Omega = \{\beta \in \mathbb{C} : \operatorname{Re} \beta > \beta_0\}$. Suppose that $a, c, b_1, \dots, b_n$ are holomorphic functions on Ω , with $\operatorname{Re} a > 0$, $\operatorname{Re} c > 0$, and $c - a = 1 - \beta$. Let $x_j \in \mathbb{C} \setminus [1, \infty)$. For $\beta_0 < \operatorname{Re} \beta < 1$, the branch of*

$$F_D^{(n)}(a(\beta); b_1(\beta), \dots, b_n(\beta); c(\beta); x_1, \dots, x_n)$$

is defined by (2.2). This branch extends holomorphically to Ω , including positive integer values of β .

PROOF. We use the logarithms of $1 - x_j t$ that vanish at $t = 0$. We set

$$\psi(t, \beta) = t^{a(\beta)-1} \prod_{j=1}^n (1 - x_j t)^{-b_j(\beta)}.$$

The function ψ is holomorphic near $t = 1$, locally uniformly in β . We expand ψ in powers of $1 - t$. On a compact subset of Ω , we fix a small $\varepsilon > 0$ and subtract sufficiently many Taylor terms from the integral over $[1 - \varepsilon, 1]$. A term $h_k(\beta)(1 - t)^k$ contributes

$$h_k(\beta) \frac{\varepsilon^{k+1-\beta}}{k+1-\beta}$$

on $[1 - \varepsilon, 1]$. The remainder integral is holomorphic after sufficiently many subtractions, and the integral over $[0, 1 - \varepsilon]$ is holomorphic because $\operatorname{Re} a > 0$. Thus the only possible poles of the unnormalized Euler integral are simple poles at $\beta = 1, 2, \dots$. The normalizing factor is

$$\frac{\Gamma(c(\beta))}{\Gamma(a(\beta))\Gamma(1-\beta)}.$$

The zeros of $1/\Gamma(1 - \beta)$ cancel all the poles of the unnormalized Euler integral. The other gamma factors are holomorphic and finite under the assumptions on a and c . $\square$

Remark 3.4. Suppose that a parametrization P_r on a real interval satisfies $\mathcal{N}_2[P_r] = Q_0 + \mu(r)R$ and agrees with a spectral factor at one endpoint of this interval. On any interval where the linear factors of P_r over $\mathbb{C}$ have the form $1 + p_j z$ with $\operatorname{Re} p_j > 0$, Lemma 3.1 determines the sign of μ' from the derivative of $\mathcal{A}_+$ or $\mathcal{A}_-$. The parametrizations below have no poles on their specified real intervals. In each case, we prove the continuation of the parametrization to the other endpoint.

4. QUINTIC PENCILS WITH TWO SQUARED QUADRATIC FACTORS

In this section, we give a two-term transformation from a rational component of the curve defined by the norm equation for a quintic pencil. We fix a real number $\sigma \geq 1$ and put

$$(4.1) \quad B = \frac{4\sigma^2(\sigma^2 + \sigma + 2)}{(\sigma + 1)^3}, \quad e = \frac{8\sigma^4}{(\sigma + 1)^3},$$

$$Q_0(v) = (1 + v)(1 + Bv + ev^2)^2 = \sum_{j=0}^5 q_j v^j, \quad \Delta = \frac{(\sigma - 1)^5}{(\sigma + 1)^3}.$$

For the pencil $Q_0 + \mu v$, we have the identity

$$(4.2) \quad Q_0(v) + \Delta v = (1 + \sigma^2 v) \left(1 + \frac{4\sigma(2\sigma^2 + \sigma + 1)}{(\sigma + 1)^3} v + \frac{e}{\sigma} v^2 \right)^2.$$

The values 0 and Δ of the pencil parameter are distinct for $\sigma > 1$. For example, at $\sigma = 2$ this identity is

$$(1 + 4v) \left(1 + \frac{88}{27}v + \frac{64}{27}v^2 \right)^2 - (1 + v) \left(1 + \frac{128}{27}v + \frac{128}{27}v^2 \right)^2 = \frac{v}{27}.$$

For $r \neq 0$, we define rational functions by

$$(4.3) \quad \begin{aligned} \tau(r) &= \frac{\sigma/r - r - 2(\sigma + 1)}{2\sigma}, & \zeta(r) &= \frac{\sigma/r + r}{2}, \\ c(r) &= \frac{e^2 \tau(r)^2 - q_4}{2e}, & d(r) &= e\tau(r), \\ b(r) &= c(r)\tau(r) - \frac{e(\tau(r) - 1)(\sigma\tau(r) - 1)\zeta(r)}{2\sqrt{2}\sigma^2}, \\ a(r) &= \frac{q_3 - c(r)^2 + 2eb(r)\tau(r)}{2e}, \\ \mu(r) &= a(r)^2 - 2b(r) - q_1, \\ P_r(z) &= 1 + a(r)z + b(r)z^2 + c(r)z^3 + d(r)z^4 + ez^5. \end{aligned}$$

We use the positive square root of 2.

We put

$$\begin{aligned} q &= \sqrt{e}, & h &= \sqrt{B + 2q}, & \tau_0 &= 1 + \frac{2h}{q}, & \omega_0 &= \sigma\tau_0 + \sigma + 1, \\ r_0 &= \frac{\sigma}{\sqrt{\omega_0^2 + \sigma + \omega_0}}, \end{aligned}$$

where all square roots are positive.

We define the constant α and the polynomial M by

$$(4.4) \quad \alpha = \frac{e(3 - 2\sqrt{2})}{128}, \quad M(r) = \frac{r^8 \mu(r)}{\alpha^2}.$$

The polynomial M has degree 16 and satisfies $M(0) = 1$. The Laurent expansion of a gives

$$a(r) = \alpha r^{-4} + O(r^{-3}) \quad (r \rightarrow 0), \quad a(r) = \frac{e(3 + 2\sqrt{2})}{128\sigma^4} r^4 + O(r^3) \quad (r \rightarrow \infty).$$

The exponents occurring in the Laurent expansion of b lie between -3 and 3 . Since $\mu = a^2 - 2b - q_1$, these two facts give the assertions on M . We fix $0 < L \leq r_0$ and set

$$(4.5) \quad \begin{aligned} E &= \mu(L), & H &= \alpha/L^4, & \eta &= L^2/\sigma, \\ W(t) &= (1 + (B-2)t + (e-B+1)t^2)^2 + Et(1-t)^4 = \prod_{i=1}^5 (1 - x_i t), \\ F_L(u) &= \frac{M(Lu) - u^8 M(L)}{1-u} = \prod_{j=1}^{15} (1 - z_j u). \end{aligned}$$

The numerator defining F_L vanishes at $u = 1$, and $F_L(0) = 1$. Both W and F_L are positive on $[0, 1]$, as shown in the proof of Theorem 4.1. We use the logarithms of the linear factors of W and F_L that vanish at the origin. The sums of these logarithms are the real logarithms of W and F_L . We use the real logarithm of H . For $\operatorname{Re} \beta \geq 1$, the target functions are the analytic continuations in β of their Euler integrals from $1/8 < \operatorname{Re} \beta < 1$ with the specified argument branches.

Then we have the following theorem.

Theorem 4.1. *Suppose $\sigma \geq 1$ and $0 < L \leq r_0$, with the notation and branches of (4.1)–(4.5). For $\operatorname{Re} \beta > 1/8$, we have*

$$(4.6) \quad \begin{aligned} & LH^{2\beta} F_D^{(5)} \left(\beta + \frac{1}{2}; \beta^{[5]}; 5\beta; x_1, \dots, x_5 \right) \\ &= \frac{\pi \Gamma(8\beta - 1) \Gamma(5\beta)}{\Gamma(\beta) \Gamma(7\beta) \Gamma(\beta + \frac{1}{2}) \Gamma(4\beta - \frac{1}{2})} \\ &\quad \times \left\{ F_D^{(15)} \left(8\beta - 1; \beta^{[15]}; 7\beta; z_1, \dots, z_{15} \right) \right. \\ &\quad \left. + \eta \frac{(8\beta - 1)_2}{(7\beta)_2} F_D^{(15)} \left(8\beta + 1; \beta^{[15]}; 7\beta + 2; z_1, \dots, z_{15} \right) \right\}. \end{aligned}$$

PROOF. *Coefficient comparison.* We write $P = 1 + az + bz^2 + cz^3 + dz^4 + ez^5$ and compute

$$(4.7) \quad \begin{aligned} \mathcal{N}_2[P](v) &= 1 + (a^2 - 2b)v + (b^2 + 2d - 2ac)v^2 \\ &\quad + (c^2 - 2bd + 2ae)v^3 + (d^2 - 2ce)v^4 + e^2v^5. \end{aligned}$$

We set $\tau = d/e$ and $Z = b - c\tau$. The coefficient equations for v^4 and v^3 determine c and a as in (4.3). The remaining coefficient equation is

$$(4.8) \quad \frac{8Z^2}{e^2} = \frac{(\tau - 1)^2(\sigma\tau - 1)^2}{\sigma^4} ((\sigma\tau + \sigma + 1)^2 + \sigma).$$

The quadratic factor on the right has distinct roots and is not a square in $\mathbb{C}(\tau)$. Thus the curve defined by the norm equation is irreducible. Its function field is the function field of the conic

$$\zeta^2 = (\sigma\tau + \sigma + 1)^2 + \sigma, \quad Z = -\frac{e(\tau - 1)(\sigma\tau - 1)\zeta}{2\sqrt{2}\sigma^2}.$$

The coordinate $r = \zeta - (\sigma\tau + \sigma + 1)$ gives the rational expressions in (4.3). The conic is nondegenerate, so the normalization of the projective closure of the curve

defined by the norm equation is $\mathbb{P}^1$. Substitution into (4.7) gives

$$\mathcal{N}_2[P_r](v) = Q_0(v) + \mu(r)v.$$

Real path. For $\mu \geq 0$, we denote the roots of $Q_0 + \mu v$ by ρ_j and choose $p_j = (-\rho_j)^{-1/2}$ with positive real parts. The product of the numbers p_j is e , since the square of this product is e^2 and the roots occur in conjugate pairs together with negative real roots. The polynomial $\prod_j (1 + p_j z)$ has real coefficients. At $\mu = 0$, it is

$$(1 + z)(1 + hz + qz^2)^2.$$

The coefficients of the displayed polynomial satisfy $d/e = \tau_0$ and $b - c\tau_0 < 0$. In fact, $b = B + 4q + 2h$, $c = B + 4q + 2hq$, and $q \geq 1$.

We put

$$\widehat{Q}(v) = e^{-2}v^5Q_0(1/v), \quad \widehat{R}(v) = e^{-2}v^4.$$

The numbers selected as in Section 3 for the pencil defined by the displayed polynomials are p_j^{-1} , and their sum is $\tau = d/e$. By Lemma 3.1 applied to this pencil, we have

$$\frac{d\tau}{d\mu} > 0, \quad \tau(\mu) - \tau_0 = \frac{1}{2\pi} \int_0^\infty v^{-3/2} \log \left(1 + \mu \frac{\widehat{R}(v)}{\widehat{Q}(v)} \right) dv.$$

Restricted to any fixed compact subinterval of $(0, \infty)$, the integral tends to infinity with μ . Thus τ increases from τ_0 to infinity. The factors $\tau - 1$ and $\sigma\tau - 1$ remain positive. Equation (4.8) therefore shows that $Z = b - c\tau$ remains negative on this path. The positive coordinate $r = \sqrt{(\sigma\tau + \sigma + 1)^2 + \sigma} - (\sigma\tau + \sigma + 1)$ decreases from r_0 to zero. Consequently, (4.3) gives the spectral factor on the whole path. On this real path, we have

$$\sum_{j=1}^5 p_j^{-1} = \frac{d(r)}{e} = \tau(r), \quad -\tau'(r) = \frac{1}{2r^2} + \frac{1}{2\sigma} > 0.$$

Thus $0 < r \leq r_0$ gives the real path, and $\mu(r)$ decreases strictly from infinity to zero. More precisely, let $\mathcal{A}_-(\mu) = \sum_j p_j^{-1}$ for $Q_0 + \mu v$. The identity $\mathcal{A}_-(\mu(r)) = \tau(r)$ and Lemma 3.1 give

$$\mu'(r) = \frac{\tau'(r)}{\mathcal{A}'_-(\mu(r))} < 0.$$

The denominator is finite and strictly positive at every finite $\mu \geq 0$. In particular, $E = \mu(L) \geq 0$ and $\mu'(L) \neq 0$.

Mellin integral. We apply (3.5) with $Q = Q_0 + Ev$ and $R = v$. The symmetric expression $\mathcal{A}_-$ on the real path is τ . For $1/8 < \operatorname{Re} \beta < 1$, we put

$$J(\beta) = \int_0^\infty v^{-1/2} \left(\frac{v}{Q_0(v) + Ev} \right)^\beta dv$$

and obtain

$$J(\beta) = 2 \sin(\pi\beta) \int_0^\infty \lambda^{-\beta} \frac{d\tau}{d\lambda} d\lambda.$$

We set $r = Lu$. Since $\mu(r)$ decreases strictly from infinity to zero on $0 < r \leq r_0$, the function $\lambda(u) = \mu(Lu) - E$ decreases from infinity to zero on $0 < u \leq 1$, and

$$\lambda(u) = H^2 u^{-8} (1 - u) F_L(u), \quad -\frac{d}{du} \tau(Lu) = \frac{1 + \eta u^2}{2Lu^2}.$$

These identities give $F_L(u) > 0$ on $[0, 1)$ and $F_L(1) = -L\mu'(L)/H^2 > 0$. Thus

$$(4.9) \quad J(\beta) = \frac{\sin(\pi\beta)}{LH^{2\beta}} \int_0^1 u^{8\beta-2}(1-u)^{-\beta} F_L(u)^{-\beta} (1+\eta u^2) du.$$

On the source side, we put $v = t/(1-t)$ and obtain

$$\begin{aligned} J(\beta) &= \int_0^1 t^{\beta-1/2} (1-t)^{4\beta-3/2} W(t)^{-\beta} dt \\ &= B(\beta + \tfrac{1}{2}, 4\beta - \tfrac{1}{2}) F_D^{(5)} \left(\beta + \tfrac{1}{2}; \beta^{[5]}; 5\beta; x_1, \dots, x_5 \right). \end{aligned}$$

The equality $W(t) = (1-t)^5(Q_0(t/(1-t)) + Et/(1-t))$ gives the positivity of W on $[0, 1)$, and $W(1) = e^2 > 0$. At $L = r_0$, we have $E = 0$, and the positivity and the nonzero endpoint derivative remain valid. The polynomial W then has degree four for $\sigma > 1$, and $W = 1$ for $\sigma = 1$. We add zero arguments according to the convention in Section 2. Thus the integral transformation includes the endpoint, at which the source curve is singular. Apply (2.2) to the two terms in (4.9). By [DLMF, (5.12.1), (5.5.3)], the beta factors satisfy

$$\frac{B(8\beta + 1, 1 - \beta)}{B(8\beta - 1, 1 - \beta)} = \frac{(8\beta - 1)_2}{(7\beta)_2},$$

and

$$\frac{\sin(\pi\beta) B(8\beta - 1, 1 - \beta)}{B(\beta + \tfrac{1}{2}, 4\beta - \tfrac{1}{2})} = \frac{\pi \Gamma(8\beta - 1) \Gamma(5\beta)}{\Gamma(\beta) \Gamma(7\beta) \Gamma(\beta + \tfrac{1}{2}) \Gamma(4\beta - \tfrac{1}{2})}.$$

This proves (4.6) for $1/8 < \operatorname{Re} \beta < 1$. The source integral is holomorphic for $\operatorname{Re} \beta > 1/8$. Since $F_L > 0$ on $[0, 1]$, none of the arguments z_j lies in $[1, \infty)$. In each target function, the parameters a and c of (2.2) satisfy $c - a = 1 - \beta$ and $\operatorname{Re} a, \operatorname{Re} c > 0$ on this half-plane. By Lemma 3.3, both target functions are holomorphic there. The gamma factors and the ratio $(8\beta - 1)_2/(7\beta)_2$ have no poles there, so the identity theorem gives (4.6) on the stated half-plane. $\square$

At $\sigma = 1$, we obtain the following specialization.

Corollary 4.2. *Suppose $\sigma = 1$, $0 < L \leq 5\sqrt{2} - 7$, and $\operatorname{Re} \beta > 1/8$. With $E, H, \eta, z_1, \dots, z_{15}$ defined by (4.5) at $B = 2$ and $e = 1$, we have*

$$\begin{aligned} LH^{2\beta} {}_5F_4 \left(\begin{matrix} \beta + \frac{1}{2}, \beta - \frac{1}{8}, \beta + \frac{1}{8}, \beta + \frac{3}{8}, \beta + \frac{5}{8} \\ \beta + \frac{1}{5}, \beta + \frac{2}{5}, \beta + \frac{3}{5}, \beta + \frac{4}{5} \end{matrix}; -\frac{256E}{3125} \right) \\ = \frac{\pi \Gamma(8\beta - 1) \Gamma(5\beta)}{\Gamma(\beta) \Gamma(7\beta) \Gamma(\beta + \tfrac{1}{2}) \Gamma(4\beta - \tfrac{1}{2})} \\ \times \left\{ F_D^{(15)} \left(8\beta - 1; \beta^{[15]}; 7\beta; z_1, \dots, z_{15} \right) \right. \\ \left. + \eta \frac{(8\beta - 1)_2}{(7\beta)_2} F_D^{(15)} \left(8\beta + 1; \beta^{[15]}; 7\beta + 2; z_1, \dots, z_{15} \right) \right\}. \end{aligned}$$

The ${}_5F_4$ function is continued from zero along the negative real axis. The other branches are those of Theorem 4.1.

PROOF. We have $Q_0(v) = (1+v)^5$ and $W(t) = 1 + Et(1-t)^4$. For $|256E/3125| < 1$, expand $W(t)^{-\beta}$ in the source integral. After division by $B(\beta + \frac{1}{2}, 4\beta - \frac{1}{2})$, the

coefficient of $(-E)^n$ is

$$\frac{(\beta)_n(\beta + \frac{1}{2})_n(4\beta - \frac{1}{2})_{4n}}{(5\beta)_{5n}n!}.$$

We split the Pochhammer symbols by the multiplication formula [DLMF, (5.5.6)] and obtain the stated ${}_5F_4$ series. The source integral gives the continuation of the ${}_5F_4$ function for $E \geq 0$. Apply Theorem 4.1. $\square$

Remark 4.3. The known Euler–Pfaff substitution [EK24, (7)] gives

$$\begin{aligned} F_D^{(5)}\left(\beta + \frac{1}{2}; \beta^{[5]}; 5\beta; x_1, \dots, x_5\right) \\ = (1 - x_5)^{-\beta-1/2} F_D^{(4)}\left(\beta + \frac{1}{2}; \beta^{[4]}; 5\beta; \frac{x_1 - x_5}{1 - x_5}, \dots, \frac{x_4 - x_5}{1 - x_5}\right). \end{aligned}$$

This is the case in which the sum of the parameters b_j of (2.2) equals c . For the branches on the right, use the image of $0 \leq t \leq 1$ under

$$u = \frac{(1 - x_5)t}{1 - x_5t}.$$

The positivity of W gives $x_5 \notin [1, \infty)$, so the denominator does not vanish for $0 \leq t \leq 1$. We put $Y_i = (x_i - x_5)/(1 - x_5)$ and transport the logarithms of the linear factors by

$$\log(1 - Y_i u) = \log(1 - x_i t) - \log(1 - x_5 t), \quad i = 1, \dots, 4.$$

The logarithm in the prefactor is the value of $\log(1 - x_5 t)$ at $t = 1$. The remaining powers in the Euler integral are transported along the same path, starting from $x_1 = \dots = x_5 = 0$ and following $x_i \mapsto s x_i$, $0 \leq s \leq 1$. This specifies the continued right-hand side even when some Y_i lies in $[1, \infty)$.

5. SEXTIC PENCILS WITH TWO FACTORIZABLE FIBERS

In this section, we construct a rational component of the curve defined by the norm equation of a sextic pencil and its real path. We fix $\sigma \geq 1$ and put

$$\begin{aligned} A &= \frac{\sigma(5\sigma^4 + 8\sigma^3 + 22\sigma^2 + 8\sigma + 5)}{(\sigma + 1)^4}, \\ B &= \frac{4\sigma^3(5\sigma^2 + 2\sigma + 5)}{(\sigma + 1)^4}, \quad F = \frac{16\sigma^5}{(\sigma + 1)^4}, \\ H(v) &= 1 + Av + Bv^2 + Fv^3, \quad \Delta = \frac{(\sigma - 1)^6}{(\sigma + 1)^4}, \\ \tilde{B}(v) &= 1 + \frac{4\sigma^2(3\sigma^2 + 2\sigma + 3)}{(\sigma + 1)^4}v + \frac{16\sigma^4}{(\sigma + 1)^4}v^2. \end{aligned} \tag{5.1}$$

For the pencil $H(v)^2 + \mu v$, we have the identity

$$H(v)^2 + \Delta v = (1 + v)(1 + \sigma^2 v)\tilde{B}(v)^2. \tag{5.2}$$

For $\sigma > 1$, the values 0 and Δ of the pencil parameter are distinct. At $\sigma = 2$, the identity is

$$\left(1 + \frac{506}{81}v + \frac{928}{81}v^2 + \frac{512}{81}v^3\right)^2 + \frac{v}{81} = (1 + v)(1 + 4v)\left(1 + \frac{304}{81}v + \frac{256}{81}v^2\right)^2.$$

We set

$$U = \frac{256\sigma^8}{(\sigma+1)^6}, \quad \nu = \frac{4096\sigma^9}{(\sigma+1)^8}, \quad q = \sqrt{\nu} > 0.$$

For $r \neq 0$, we define rational functions by

$$(5.3) \quad \begin{aligned} e(r) &= \frac{q}{2\sqrt{5}}(r^{-1} - r), & w(r) &= \frac{q}{2}(r^{-1} + r), \\ c(r) &= \frac{-e(r)(e(r)^2 + 4BF) + (e(r)^2 - U)w(r)}{8F^2}, \\ d(r) &= \frac{e(r)^2 - 2BF}{2F}, & b(r) &= \frac{B^2 + 2AF - d(r)^2 + 2c(r)e(r)}{2F}, \\ a(r) &= \frac{4F + 2AB - c(r)^2 + 2b(r)d(r)}{2e(r)}, \\ \mu(r) &= a(r)^2 - 2b(r) - 2A, \\ P_r(z) &= 1 + a(r)z + b(r)z^2 + c(r)z^3 + d(r)z^4 + e(r)z^5 + Fz^6. \end{aligned}$$

We extend these functions across their removable singularities. All square roots of constants in this section are positive.

We define

$$(5.4) \quad \begin{aligned} D_\sigma &= 5\sigma^2 + 6\sigma + 5, & Z &= \frac{\sqrt{5}}{4\sqrt{\sigma}}(\sigma + 1 + \sqrt{D_\sigma}), \\ r_* &= \frac{1}{\sqrt{1 + Z^2} + Z}, & D_0 &= \frac{2}{\sqrt{5}\sigma}, & \alpha &= \frac{8(25 - 11\sqrt{5})\sigma^{5/2}}{125(\sigma + 1)^4}. \end{aligned}$$

We denote the spectral factor of $H(-z^2)$, with constant term one, by

$$S_\sigma(z) = 1 + a_1z + b_1z^2 + fz^3, \quad f = \sqrt{F} > 0.$$

We have $a_1, b_1 > 0$, since the real linear factors and the real quadratic factors of S_σ have positive coefficients. We put

$$e_0 = 2b_1f, \quad r_{\text{start}} = \frac{\sqrt{5e_0^2 + \nu} - \sqrt{5}e_0}{q} = \frac{q}{\sqrt{5e_0^2 + \nu} + \sqrt{5}e_0}.$$

The endpoint r_{start} corresponds to the polynomial H^2 at $\mu = 0$. The point r_* corresponds to the factorized fiber in (5.2). In the proof of Theorem 5.1, we show that $r_{\text{start}} > r_*$ for $\sigma > 1$ and $r_{\text{start}} = r_*$ for $\sigma = 1$.

The functions a, c, e are odd Laurent polynomials in r , and b, d are even Laurent polynomials. By $U^2\nu = 256F^5$, we see that the apparent pole at $e = 0$ in the expression for a cancels. The Laurent expansion of a gives

$$\begin{aligned} a(r) &= \alpha r^{-5} + O(r^{-3}) \quad (r \rightarrow 0), \\ a(r) &= \frac{8(25 + 11\sqrt{5})\sigma^{5/2}}{125(\sigma + 1)^4} r^5 + O(r^3) \quad (r \rightarrow \infty). \end{aligned}$$

The exponents occurring in the Laurent expansion of b lie between -4 and 4 . Consequently

$$(5.5) \quad M(u) = \frac{u^5 \mu(\sqrt{u})}{\alpha^2}$$

is a polynomial of degree ten with $M(0) = 1$. The polynomial M does not depend on the choice of $\sqrt{u}$. We fix $0 < L \leq r_{\text{start}}$ and put

$$(5.6) \quad \begin{aligned} E &= \mu(L) \geq 0, & C &= \alpha/L^5, & Q(v) &= H(v)^2 + Ev, \\ T(t) &= (1-t)^3 H\left(\frac{t}{1-t}\right), & W(t) &= T(t)^2 + Et(1-t)^5 = \prod_{i=1}^6 (1-x_i t), \\ V_L(t) &= \frac{M(L^2 t) - t^5 M(L^2)}{1-t} = \prod_{j=1}^9 (1-z_j t). \end{aligned}$$

Both W and V_L are positive on $[0, 1]$, as shown in the proof of Theorem 5.1. We use the logarithms of the linear factors of W and V_L that vanish at the origin. The sums of these logarithms are the real logarithms of W and V_L . We use the real logarithm of C . For $\text{Re } \beta \geq 1$, the target functions are the analytic continuations in β of their Euler integrals from $1/10 < \text{Re } \beta < 1$ with the specified argument branches.

Then we have the following theorem.

Theorem 5.1. *Suppose $\sigma \geq 1$ and $0 < L \leq r_{\text{start}}$, with the notation and branches of (5.1)–(5.6). For $\text{Re } \beta > 1/10$, we have*

$$(5.7) \quad \begin{aligned} & \frac{LC^{2\beta}}{D_0} F_D^{(6)} \left(\beta + \frac{1}{2}; \beta^{[6]}; 6\beta; x_1, \dots, x_6 \right) \\ &= \frac{\pi \Gamma(6\beta)}{\Gamma(\beta) \Gamma(\beta + \frac{1}{2}) \Gamma(4\beta + \frac{1}{2})} \\ & \times \left\{ F_D^{(9)} \left(5\beta - \frac{1}{2}; \beta^{[9]}; 4\beta + \frac{1}{2}; z_1, \dots, z_9 \right) \right. \\ & \quad \left. + L^2 \frac{5\beta - \frac{1}{2}}{4\beta + \frac{1}{2}} F_D^{(9)} \left(5\beta + \frac{1}{2}; \beta^{[9]}; 4\beta + \frac{3}{2}; z_1, \dots, z_9 \right) \right\}. \end{aligned}$$

PROOF. *Coefficient comparison.* We write $P = 1 + az + bz^2 + cz^3 + dz^4 + ez^5 + Fz^6$ and compute

$$\begin{aligned} \mathcal{N}_2[P](v) &= 1 + (a^2 - 2b)v + (b^2 + 2d - 2ac)v^2 \\ & \quad + (c^2 + 2ae - 2bd - 2F)v^3 + (d^2 + 2bF - 2ce)v^4 \\ & \quad + (e^2 - 2dF)v^5 + F^2v^6. \end{aligned}$$

The coefficients of v^5, v^4, v^3 determine d, b, a in that order when $e \neq 0$. After substitution, the coefficient equation for v^2 is equivalent to

$$(5.8) \quad (4BF e + 4F^2 c - e^3)G(c, e) = 0,$$

where

$$G(c, e) = 16F^4 c^2 + (16BF^3 e + 4F^2 e^3)c + 16AF^3 e^2 + 4BF e^4 - 64F^5 - e^6.$$

As a quadratic polynomial in c , G has the discriminant

$$16F^4 \{5e^6 - 8BF e^4 + 16F^2 (B^2 - 4AF) e^2 + 256F^5\}.$$

For the constants A, B, F of (5.1), the expression in braces equals

$$(e^2 - U)^2 (5e^2 + \nu).$$

Since $\nu > 0$, the polynomial $5e^2 + \nu$ has distinct roots. Thus G is irreducible over $\mathbb{C}(e)$, and the function field of G is that of $w^2 = 5e^2 + \nu$. Solving the quadratic equation for c gives the expression in (5.3). The coordinate $r = (w - \sqrt{5}e)/q$ gives the displayed expressions for e and w . Thus the normalization of the projective closure of this irreducible component has genus zero. Substitution gives

$$\mathcal{N}_2[P_r](v) = H(v)^2 + \mu(r)v.$$

Real path. We set

$$g = \frac{2\sigma\sqrt{D_\sigma}}{(\sigma+1)^2}, \quad k = \frac{4\sigma^2}{(\sigma+1)^2}.$$

At $\mu = \Delta$, the spectral factor is

$$(5.9) \quad P_*(z) = (1+z)(1+\sigma z)(1+gz+kz^2)^2.$$

The linear factors of P_* over $\mathbb{C}$ have the form $1+p_jz$ with $\operatorname{Re} p_j > 0$, since $g, k > 0$. The coefficient of z^5 in P_* is

$$e_* = \frac{16\sigma^4}{(\sigma+1)^4}(\sigma+1+\sqrt{D_\sigma}) > \sqrt{U}.$$

Substitute (5.4) into (5.3) to obtain (5.9). For this substitution, we use

$$\frac{e(r_*)}{F} = \frac{\sigma+1+\sqrt{D_\sigma}}{\sigma}, \quad w(r_*) = \frac{16\sigma^4(5\sigma+5+\sqrt{D_\sigma})}{(\sigma+1)^4}.$$

For every $\mu \geq 0$, the polynomial $H^2 + \mu v$ is positive on $[0, \infty)$, so its substitution $v = -z^2$ admits a spectral factor. At $\mu = 0$, this spectral factor is S_σ^2 . We write $S_\sigma = (1+pz)(1+hz+k_1z^2)$, with $p, h, k_1 > 0$. This factorization also includes the case in which all three zeros of S_σ are real. Since $a_1 = p+h$, $b_1 = k_1+ph$, and $f = pk_1$, we have

$$a_1b_1 - f = h(k_1+p^2+ph) > 0.$$

The factor $4BF e + 4F^2 c - e^3$ in (5.8) has the value $8f^4(f - a_1b_1) \neq 0$ at $\mu = 0$. Hence $G = 0$ holds near $\mu = 0$ and, by analyticity of the coefficients of the spectral factor, for every $\mu \geq 0$.

The identity $\mathcal{N}_2[S_\sigma] = H$ gives

$$a_1^2 - 2b_1 = A, \quad b_1^2 - 2a_1f = B.$$

Consequently,

$$e_0^2 = 4BF + 8a_1f^3 > 4BF, \quad \frac{4BF}{U} = 1 + \frac{4(\sigma^2+1)}{(\sigma+1)^2} > 1.$$

Thus $e_0 > \sqrt{U}$, including the case $\sigma = 1$.

We denote the coefficient of z^5 on the real path by $e(\mu)$. By (3.3), we have

$$\frac{e'(\mu)}{F} = \frac{1}{2\pi} \int_0^\infty \frac{v^{1/2}}{H(v)^2 + \mu v} dv > 0 \quad (\mu \geq 0).$$

This integral is finite, including at $\mu = 0$. Integration of this kernel gives

$$\frac{e(\mu) - e_0}{F} = \frac{1}{2\pi} \int_0^\infty v^{-1/2} \log \left(\frac{H(v)^2 + \mu v}{H(v)^2} \right) dv.$$

Integrating the same kernel from Δ , we obtain

$$\frac{e(\mu) - e_*}{F} = \frac{1}{2\pi} \int_0^\infty v^{-1/2} \log \left(\frac{H(v)^2 + \mu v}{H(v)^2 + \Delta v} \right) dv.$$

Restricted to a fixed compact subinterval of $(0, \infty)$, the integral tends to infinity as μ tends to infinity. Thus $e(\mu)$ increases from e_0 to infinity as μ ranges over $[0, \infty)$.

For $e \geq e_0 > \sqrt{U}$, the two roots of G as a polynomial in c are distinct. The root with the negative sign in (5.3) is negative. The coefficients of a real polynomial with constant term one and all zeros in the left half-plane are positive, since its real linear factors and real quadratic factors have positive coefficients. Hence the real path has the positive sign in (5.3) for every $\mu \geq 0$. On this branch, we have $w = \sqrt{5e^2 + \nu}$, and

$$r = \frac{w - \sqrt{5}e}{q} > 0, \quad \frac{dr}{de} = -\frac{\sqrt{5}r}{w} < 0.$$

As μ ranges from zero to infinity, this coordinate decreases from r_{start} to zero. Thus $0 < r \leq r_{\text{start}}$ gives the whole real path for $\mu \geq 0$, and

$$\mu(r_{\text{start}}) = 0, \quad \lim_{r \downarrow 0} \mu(r) = \infty.$$

Since $\mu(r_*) = \Delta$, we have $r_{\text{start}} > r_*$ for $\sigma > 1$. At $\sigma = 1$, we have $S_1 = (1 + z)^3$, $e_0 = 6$, and

$$r_{\text{start}} = r_* = \frac{7 - 3\sqrt{5}}{2}.$$

The symmetric expression $\mathcal{A}_-$ and its derivative are

$$\frac{e(r)}{F} = D_0(r^{-1} - r), \quad -\frac{d}{dr} \frac{e(r)}{F} = D_0(r^{-2} + 1) > 0.$$

Let $\mathcal{A}_-(\mu) = \sum_j p_j^{-1}$ for $H^2 + \mu v$. Differentiating $\mathcal{A}_-(\mu(r)) = e(r)/F$ gives

$$\mu'(r) = \frac{-D_0(r^{-2} + 1)}{\mathcal{A}'_-(\mu(r))} < 0,$$

because the denominator is finite and strictly positive by Lemma 3.1. This formula also holds at $r = r_{\text{start}}$. Thus $0 < L \leq r_{\text{start}}$ gives $E = \mu(L) \geq 0$ and $\mu'(L) \neq 0$.

Mellin integral. We apply (3.5) with $R = v$ and $Q = H^2 + Ev$. For $1/10 < \text{Re } \beta < 1$, we put $r = L\sqrt{t}$. Then

$$\lambda(t) = \mu(L\sqrt{t}) - E = C^2 t^{-5} (1 - t) V_L(t), \quad -\frac{d}{dt} \frac{e(L\sqrt{t})}{F} = \frac{D_0}{2L} t^{-3/2} (1 + L^2 t).$$

The function λ decreases from infinity to zero. Thus $V_L(t) > 0$ for $0 \leq t < 1$ and $V_L(1) = -L\mu'(L)/(2C^2) > 0$. The source integral becomes

$$\begin{aligned} & \int_0^\infty v^{-1/2} \left(\frac{v}{Q(v)} \right)^\beta dv \\ &= \frac{D_0}{LC^{2\beta}} \sin(\pi\beta) \int_0^1 t^{5\beta-3/2} (1-t)^{-\beta} V_L(t)^{-\beta} (1+L^2 t) dt. \end{aligned}$$

On the other hand, $v = t/(1-t)$ gives

$$\begin{aligned} & \int_0^\infty v^{-1/2} \left(\frac{v}{Q(v)} \right)^\beta dv \\ &= B(\beta + \tfrac{1}{2}, 5\beta - \tfrac{1}{2}) F_D^{(6)} \left(\beta + \tfrac{1}{2}; \beta^{[6]}; 6\beta; x_1, \dots, x_6 \right). \end{aligned}$$

The polynomial W is positive on $[0, 1)$ by the positivity of Q , and $W(1) = F^2 > 0$. Apply (2.2) to the two terms of the target integral. By [DLMF, (5.12.1), (5.5.3)], the ratio of their beta factors is $(5\beta - \frac{1}{2})/(4\beta + \frac{1}{2})$, and

$$\frac{\sin(\pi\beta)B(5\beta - \frac{1}{2}, 1 - \beta)}{B(\beta + \frac{1}{2}, 5\beta - \frac{1}{2})} = \frac{\pi\Gamma(6\beta)}{\Gamma(\beta)\Gamma(\beta + \frac{1}{2})\Gamma(4\beta + \frac{1}{2})}.$$

This proves (5.7) for $1/10 < \operatorname{Re}\beta < 1$. The source integral is holomorphic for $\operatorname{Re}\beta > 1/10$. Since $V_L > 0$ on $[0, 1]$, none of the arguments z_j lies in $[1, \infty)$. Both target functions satisfy the assumptions of Lemma 3.3 on this half-plane. The gamma factors and the ratio $(5\beta - \frac{1}{2})/(4\beta + \frac{1}{2})$ also have no poles there. The identity theorem gives (5.7) on the stated half-plane. $\square$

At $\sigma = 1$, we obtain the following specialization.

Corollary 5.2. *Under the assumptions of Theorem 5.1 with $\sigma = 1$, we have*

$$\begin{aligned} & \frac{LC^{2\beta}}{D_0} {}_5F_4 \left(\begin{matrix} \beta - \frac{1}{10}, \beta + \frac{1}{10}, \beta + \frac{3}{10}, \beta + \frac{1}{2}, \beta + \frac{7}{10} \\ \beta + \frac{1}{6}, \beta + \frac{1}{3}, \beta + \frac{2}{3}, \beta + \frac{5}{6} \end{matrix}; -\frac{3125E}{46656} \right) \\ &= \frac{\pi\Gamma(6\beta)}{\Gamma(\beta)\Gamma(\beta + \frac{1}{2})\Gamma(4\beta + \frac{1}{2})} \\ & \times \left\{ F_D^{(9)} \left(5\beta - \frac{1}{2}; \beta^{[9]}; 4\beta + \frac{1}{2}; z_1, \dots, z_9 \right) \right. \\ & \quad \left. + L^2 \frac{5\beta - \frac{1}{2}}{4\beta + \frac{1}{2}} F_D^{(9)} \left(5\beta + \frac{1}{2}; \beta^{[9]}; 4\beta + \frac{3}{2}; z_1, \dots, z_9 \right) \right\}. \end{aligned}$$

Here we specialize the polynomials determining the arguments and the constants at $\sigma = 1$. We continue the ${}_5F_4$ function from zero along the negative real axis.

PROOF. We have $H(v) = (1+v)^3$ and $W(t) = 1 + Et(1-t)^5$. Expanding the source integral near $E = 0$ and dividing by its beta factor, we obtain the general term

$$\frac{(-E)^n}{n!} \frac{(\beta)_n (\beta + \frac{1}{2})_n (5\beta - \frac{1}{2})_{5n}}{(6\beta)_{6n}}.$$

By the multiplication formula [DLMF, (5.5.6)], the factors $(\beta)_n$ and $(\beta + \frac{1}{2})_n$ cancel against the denominator. This cancellation gives the stated ${}_5F_4$ series. Continue along $E \geq 0$ and apply Theorem 5.1. $\square$

We use additional upper parameters equal to -1 to represent the polynomial factors in the target integrals.

Remark 5.3. Under the assumptions and branches of Theorem 4.1, we put $\eta = L^2/\sigma > 0$. Fact 2.2 gives

$$\begin{aligned} & F_D^{(15)} \left(8\beta - 1; \beta^{[15]}; 7\beta; z_1, \dots, z_{15} \right) \\ & + \eta \frac{(8\beta - 1)_2}{(7\beta)_2} F_D^{(15)} \left(8\beta + 1; \beta^{[15]}; 7\beta + 2; z_1, \dots, z_{15} \right) \\ (5.10) \quad & = F_D^{(17)} \left(8\beta - 1; \beta^{[15]}, -1, -1; 7\beta; z_1, \dots, z_{15}, \sqrt{-1}\sqrt{\eta}, -\sqrt{-1}\sqrt{\eta} \right). \end{aligned}$$

The two added factors have product $1 + \eta u^2$ on the integration interval. Under the assumptions and branches of Theorem 5.1, we have

$$F_D^{(9)} \left(5\beta - \frac{1}{2}; \beta^{[9]}; 4\beta + \frac{1}{2}; z_1, \dots, z_9 \right)$$

$$\begin{aligned}
& + L^2 \frac{5\beta - \frac{1}{2}}{4\beta + \frac{1}{2}} F_D^{(9)} \left(5\beta + \frac{1}{2}; \beta^{[9]}; 4\beta + \frac{3}{2}; z_1, \dots, z_9 \right) \\
(5.11) \quad & = F_D^{(10)} \left(5\beta - \frac{1}{2}; \beta^{[9]}, -1; 4\beta + \frac{1}{2}; z_1, \dots, z_9, -L^2 \right).
\end{aligned}$$

The added factor is $1 + L^2 t$. Both equalities hold first in the strips in which the Euler integrals converge and then on the half-planes of the respective theorems by Lemma 3.3.

6. NORM EQUATIONS AND HYPERELLIPTIC PERIODS

In this section, we determine the normalizations of the curves defined by the norm equations and construct the correspondences of their hyperelliptic integrals.

6.1. Normalization of the curves defined by the norm equation

In this subsection, we determine the normalization of the two selected curves defined by the norm equation. We use the notation of (4.1) and (5.1) in the corresponding cases. Then we have the following proposition.

Proposition 6.1. *Suppose that $\sigma \geq 1$. The projective closure of the quintic curve defined by the norm equation has normalization isomorphic to $\mathbb{P}^1$. For the sextic component $G = 0$ in (5.8), the normalization of its projective closure is isomorphic to $\mathbb{P}^1$. Before specializing the coefficients to the two families, we obtain double covers by eliminating the other coefficients. Each cover has genus two when its defining polynomial of degree six is squarefree. In the displayed families, we remove the squared factors and obtain the conics*

$$(6.1) \quad \zeta^2 = (\sigma\tau + \sigma + 1)^2 + \sigma, \quad w^2 = 5e^2 + \nu.$$

Here $\tau = d/e$ in the quintic case, and e is the coefficient of z^5 in the sextic case. By using the rational coordinates of the corresponding main theorems, we have

$$\begin{aligned}
(6.2) \quad \mathcal{A}_-(\mu(r)) &= \frac{\sigma/r - r - 2(\sigma + 1)}{2\sigma}, & -\frac{d}{dr} \mathcal{A}_-(\mu(r)) &= \frac{1}{2r^2} + \frac{1}{2\sigma}, \\
\mathcal{A}_-(\mu(r)) &= D_0(r^{-1} - r), & -\frac{d}{dr} \mathcal{A}_-(\mu(r)) &= D_0(r^{-2} + 1),
\end{aligned}$$

where the first row is for the quintic and the second row is for the sextic.

PROOF. We first keep the quintic coefficients q_2, q_3, q_4, e independent and put $Z = b - cd/e$. By eliminating c and a from the coefficient equations in (4.7), we obtain

$$\begin{aligned}
8e^4 Z^2 &= D_5(d), & D_5(d) &= d^6 - q_4 d^4 + (4e^2 q_3 - q_4^2) d^2 - 16e^4 d \\
&& &+ q_4^3 - 4e^2 q_3 q_4 + 8e^4 q_2.
\end{aligned}$$

If D_5 is squarefree, the double cover has six branch points. We use the coordinate $\tau = d/e$ on $\mathbb{P}^1$. An invariant regular differential under $Z \mapsto -Z$ descends to $\mathbb{P}^1$ and hence vanishes. Every regular differential therefore has the form $\phi(\tau) d\tau/Z$ with $\phi \in \mathbb{C}(\tau)$. Regularity at the finite points forces ϕ to be a polynomial, and regularity at infinity gives $\deg \phi \leq 1$. Thus $d\tau/Z$ and $\tau d\tau/Z$ form a basis, and the genus is two. For the coefficients in (4.1), we have

$$e^{-6} D_5(e\tau) = \frac{(\tau - 1)^2 (\sigma\tau - 1)^2}{\sigma^4} ((\sigma\tau + \sigma + 1)^2 + \sigma).$$

Removing the squared factors in the function field gives the first conic in (6.1). The discriminant of its quadratic polynomial in τ is $-4\sigma^3 \neq 0$.

For the sextic component, we keep A, B, F independent and set

$$D_6(e) = 5e^6 - 8BFe^4 + 16F^2(B^2 - 4AF)e^2 + 256F^5.$$

The discriminant of G with respect to c is $16F^4D_6(e)$. If D_6 is squarefree, the double cover again has six branch points and genus two. For the coefficients in (5.1), we have

$$D_6(e) = (e^2 - U)^2(5e^2 + \nu).$$

Removing the squared factor gives the second conic in (6.1). The discriminant of its quadratic polynomial in e is $-20\nu \neq 0$.

For the first conic, we put $r = \zeta - (\sigma\tau + \sigma + 1)$. For the second conic, we put $r = (w - \sqrt{5}e)/\sqrt{\nu}$. These coordinates give the parametrizations in the two main theorems. On the real paths, the coefficient ratios d/e and e/F equal the sums of the reciprocals of the numbers p_j in the quintic and sextic cases, respectively. Differentiating these ratios gives (6.2). $\square$

Remark 6.2. The normalization calculation concerns the curve defined by the norm equation and its selected component. The proofs of Theorems 4.1 and 5.1 use Lemma 3.1 to continue the real paths along the stated real intervals. On these intervals, we have

$$\mu'(r) = \frac{\frac{d}{dr}\mathcal{A}_-(\mu(r))}{\mathcal{A}'_-(\mu(r))} < 0.$$

The denominator is finite and positive, including pencil values with multiple roots. Thus the map $r \mapsto \mu(r)$ has no critical point on either selected real interval. The displayed genus-two models do not assert that every polynomial pencil satisfying $Q_\mu(v) > 0$ for $v \geq 0$ admits such a rational coordinate.

6.2. Changes of the variable and the generators of the pencils

In this subsection, we compare the degree-five and degree-six pencils at $\sigma > 1$ with their specializations at $\sigma = 1$, including the fibers at infinity. We homogenize each pencil to its fixed degree. After homogenization, we allow a projective change of the variable and an invertible linear change of the two generators. Then we have the following proposition.

Proposition 6.3. *For every $\sigma > 1$, the quintic pencil in (4.1) cannot be identified with $(1+v)^5 + \mu v$ by a projective change of the variable together with an invertible linear change of the two generators. For every $\sigma > 1$, the sextic pencil in (5.1) cannot be identified with $(1+v)^6 + \mu v$ by a projective change of the variable together with an invertible linear change of the two generators.*

PROOF. For the quintic coefficients, we compute

$$B^2 - 4e = \frac{16\sigma^4(\sigma - 1)^2(\sigma^2 + 2\sigma + 2)}{(\sigma + 1)^6} > 0,$$

$$1 - B + e = \frac{(\sigma - 1)^2(4\sigma + 1)}{(\sigma + 1)^2} > 0.$$

The quadratic factor $1 + Bv + ev^2$ of Q_0 has two distinct negative roots, neither equal to -1 . Thus the fiber $\mu = 0$ has root multiplicities $2 + 2 + 1$.

For the sextic coefficients, we compute

$$\text{disc}_v H = \frac{16\sigma^8(\sigma - 1)^6(125\sigma^4 + 400\sigma^3 + 678\sigma^2 + 400\sigma + 125)}{(\sigma + 1)^{14}} > 0.$$

The cubic H has three distinct real roots. Since its coefficients are positive, all three roots are negative. Thus the fiber $\mu = 0$ of $H^2 + \mu v$ has root multiplicities $2 + 2 + 2$.

For $N = 5, 6$, we put $P_N(v, \mu) = (1 + v)^N + \mu v$. A multiple finite root is nonzero and satisfies

$$v\partial_v P_N - P_N = (1 + v)^{N-1}((N - 1)v - 1) = 0.$$

At $v = -1$, the pencil value is $\mu = 0$, and the fiber has one root of multiplicity N . At $v = 1/(N - 1)$, the pencil value is $-N^N/(N - 1)^{N-1}$. The second derivative is nonzero there, so this fiber has one double root and $N - 2$ simple roots. The fiber at infinity is the homogenization of v , with multiplicities $(N - 1) + 1$. Every other fiber has N simple roots. Neither $2 + 2 + 1$ for $N = 5$ nor $2 + 2 + 2$ for $N = 6$ occurs in this list. Projective changes of the variable preserve root multiplicities, and changes of the two generators permute the fibers. Hence neither pair of pencils can be identified by the stated changes. $\square$

Remark 6.4. Proposition 6.3 concerns only the changes of the variable and the two generators specified before its statement, applied to the homogenized pencils. It does not classify algebraic substitutions of higher degree or transformations of the associated hypergeometric functions.

6.3. A correspondence from the norm identity

In this subsection, we express the trace of a differential by power sums of the roots on a curve defined by the norm equation. We fix a smooth connected projective curve $\mathcal{B}$ over $\mathbb{C}$ and put $K = \mathbb{C}(\mathcal{B})$. We fix $Q_0 \in \mathbb{C}[v]$ of degree $N \geq 2$, with $Q_0(0) = 1$, and a nonconstant function $\mu \in K$. We fix $P(z) \in K[z]$ of degree N , with $P(0) = 1$, satisfying

$$(6.3) \quad P(z)P(-z) = Q_0(-z^2) - \mu z^2.$$

For every integer $m \geq 1$, we define the power sum $s_m \in K$ by summing z^m over all N roots of P , with multiplicities. Then we have the following proposition.

Proposition 6.5. *Let $E \in \mathbb{C}$ satisfy the following conditions: $Q_E(v) = Q_0(v) + Ev$ is squarefree, and $\mu - E$ is not a square in K . Let $\mathcal{C}$ and $\mathcal{T}$ be the smooth projective models of*

$$y^2 = Q_E(v), \quad t^2 = \mu - E,$$

respectively. Let $\mathcal{D}$ be the disjoint union of the normalized projective components of the reduced curve defined by $P(z) = 0$ that dominate $\mathcal{T}$. The projection $\pi : \mathcal{D} \rightarrow \mathcal{T}$ has total degree N . The formulas

$$v = -z^2, \quad y = zt$$

give a morphism $f : \mathcal{D} \rightarrow \mathcal{C}$, whose restriction to every component is dominant. For every integer $k \geq 0$, we have the identity of meromorphic differentials

$$(6.4) \quad \text{Tr}_\pi f^* \left(\frac{v^k dv}{y} \right) = \frac{2(-1)^{k+1}}{(2k + 1)t} ds_{2k+1}.$$

For $0 \leq k < g = \lfloor (N-1)/2 \rfloor$, the differentials on $\mathcal{C}$ form a basis of its regular differentials, and their traces are regular on $\mathcal{T}$. These traces change sign under the involution $t \mapsto -t$ of $\mathcal{T}$.

PROOF. We put $F(z) = Q_0(-z^2) - \mu z^2$. A repeated nonzero root of F would determine a constant pencil value from $vQ'_0(v) - Q_0(v) = 0$. The polynomial $vQ'_0(v) - Q_0(v)$ has constant term -1 , so there are only finitely many such values. Since μ is nonconstant, the polynomial F is separable over K . The identity (6.3) implies that P is separable. Thus the reduced curve defined by $P(z) = 0$ has total degree N over $\mathcal{T}$. On the curve defined by $P(z) = 0$, we have

$$Q_E(-z^2) = Q_0(-z^2) - Ez^2 = (\mu - E)z^2 = (zt)^2.$$

A constant root z is nonzero because $P(0) = 1$. The norm identity would then make μ constant. Hence $v = -z^2$ is nonconstant on every component. The rational maps to $\mathcal{C}$ extend to the projective normalizations, and each component dominates $\mathcal{C}$.

On each component of $\mathcal{D}$, we compute

$$f^* \left(\frac{v^k dv}{y} \right) = \frac{2(-1)^{k+1}}{t} z^{2k} dz = \frac{2(-1)^{k+1}}{(2k+1)t} d(z^{2k+1}).$$

The field trace commutes with d in characteristic zero. Taking the sum over all components of the correspondence proves (6.4). The trace of a regular differential under a finite morphism is regular. In local coordinates $u = s^e$, summing a regular expansion in s keeps only terms $s^{e\ell-1} ds$, which become multiples of $u^{\ell-1} du$, with $\ell \geq 1$. The source forms are regular at finite points. At infinity, their orders are $2g-2-2k$ for odd N and $g-1-k$ at both points for even N . The defining polynomial of $\mathcal{C}$ has degree N . Thus the curve $\mathcal{C}$ has genus g , and these g independent forms give a basis. Since each s_m comes from K , changing the sign of t changes the sign of (6.4). $\square$

Corollary 6.6. *Under the assumptions of Proposition 6.5, let $b \in \mathcal{B}$ satisfy $\mu(b) = E$. The coefficients and the roots of P extend holomorphically near b . For every $m \geq 1$, the differential ds_m/t is regular at every point of $\mathcal{T}$ over b .*

PROOF. Comparing leading coefficients in (6.3) shows that the leading coefficient of P is constant and nonzero. Each root of P is integral over the local ring at b , since the polynomial on the right-hand side of (6.3) has a constant nonzero leading coefficient. The coefficients of P are therefore integral and belong to K , so they extend across b , since $\mathcal{B}$ is smooth. The polynomial $Q_E(-z^2)$ has distinct nonzero roots, which gives local holomorphic root branches. Differentiating their norm equations and using $d\mu = 2tdt$, we obtain

$$dz_j = -\frac{z_j}{2\{Q'_0(-z_j^2) + \mu\}} d\mu, \quad \frac{ds_m}{t} = -m \sum_j \frac{z_j^m}{Q'_0(-z_j^2) + \mu} dt.$$

The denominators are nonzero at $\mu = E$. The second expression is regular at every zero of t on the normalization. $\square$

Remark 6.7. For a multiple zero of $\mu - E$, the function t need not be a local parameter.

Remark 6.8. The rank of the regular traces requires a separate computation. For example, we take

$$P_r(z) = 1 + rz + z^4, \quad Q_0(v) = (1 + v^2)^2, \quad \mu = r^2, \quad E = 1.$$

The source has genus one, the target $t^2 = r^2 - 1$ has genus zero, and $s_1 = 0$. Every component of the correspondence dominates the source, but the only regular source differential has zero trace.

6.4. A condition for rational curves defined by the norm equation

In this subsection, we give a sufficient condition for a component of the curve defined by the norm equation to be rational, assuming that its function field has the form stated in the following lemma. Then we have the following lemma.

Lemma 6.9. *Suppose that a component of the curve defined by the norm equation has function field $\mathbb{C}(x)(\sqrt{D(x)})$, where $D \in \mathbb{C}[x]$ is nonzero. We write $D = A^2 D_{\text{sf}}$, where D_{sf} retains precisely the factors of odd multiplicity. If $\deg D_{\text{sf}} \leq 2$, the normalization of the component is rational.*

PROOF. We replace $\sqrt{D}$ by $A\sqrt{D_{\text{sf}}}$ in the function field. If D_{sf} is constant, the component has function field $\mathbb{C}(x)$. If the degree of D_{sf} is one or two, we obtain a nondegenerate conic over $\mathbb{C}$. Each case gives a rational normalization. $\square$

Remark 6.10. This condition applies when the function field of the selected component has the form stated in Lemma 6.9. For a real Mellin integral, we separately identify a real interval on which P is the spectral factor. We also prove the range and orientation of μ and its nonzero derivative at each finite endpoint. For example, $P_r = 1 + rz + z^2$ has positive norm $1 + (r^2 - 2)v + v^2$ on $v \geq 0$ for $r \leq -\sqrt{2}$. The roots of P_r lie in the right half-plane, so this rational curve defined by the norm equation does not select a real branch on which all zeros of P_r lie in the left half-plane.

6.5. Correspondences and two Abelian integrals

In this subsection, we construct algebraic correspondences for the exponent $\beta = 1/2$ and compute the traces of two regular differentials. We use the quintic or sextic pencil and its notation from Theorem 4.1 or Theorem 5.1, respectively. For the algebraic correspondence, we now fix $E \in \mathbb{C}$. We return to the real paths in Corollary 6.25. We put

$$C_E: \quad y^2 = Q_E(v), \quad Q_E(v) = Q_0(v) + Ev,$$

where $Q_0 = H^2$ in the sextic case. For the quintic and sextic cases, respectively, we put

$$\begin{aligned} S_E(r) &= M(r) - Er^8/\alpha^2, & T_E: \quad \xi^2 &= S_E(r), \\ S_E(x) &= M(x) - Ex^5/\alpha^2, & T_E: \quad \xi^2 &= S_E(x). \end{aligned}$$

We use the smooth projective models of these curves when Q_E and S_E are square-free. For each fixed $\sigma \geq 1$, only finitely many values of E violate this condition. In fact, a multiple root of Q_E satisfies $vQ'_0(v) - Q_0(v) = 0$, whose constant term is -1 . A multiple root of S_E is a critical point of the nonconstant rational function μ , away from its poles.

For each pencil, we write $z_1(r), \dots, z_N(r)$ for the roots of P_r and put $s_3(r) = \sum_j z_j(r)^3$. Here $z_j(r)$ are roots of P_r , whereas z_j in the transformation formulas are Lauricella arguments. We define the polynomials B_5 and B_6 by

$$B_5(r) = \frac{2r^4}{3\alpha} \frac{ds_3}{dr}, \quad B_6(r^2) = \frac{r^4}{3\alpha} \frac{ds_3}{dr}$$

for the quintic and sextic pencils, respectively. The second expression is even in r . The coefficient expansions following the theorem show that these expressions are polynomials in r and $x = r^2$, respectively.

Then we have the following theorem.

Theorem 6.11. *Suppose that $\sigma \geq 1$, and that Q_E and S_E are squarefree. For the quintic pencil, there is an algebraic correspondence*

$$C_E \xleftarrow{p} D_E \xrightarrow{q} T_E$$

with $g(C_E) = 2$, $g(T_E) = 7$, $\deg p = 16$, and $\deg q = 5$, whose traces satisfy

$$q_* p^* \left(\frac{dv}{y} \right) = -\frac{r^2 + r^4/\sigma}{\alpha\xi} dr, \quad q_* p^* \left(\frac{v dv}{y} \right) = B_5(r) \frac{dr}{\xi}.$$

For the sextic pencil, there is an algebraic correspondence of the same form with $g(C_E) = 2$, $g(T_E) = 4$, $\deg p = 10$, and $\deg q = 6$, whose traces satisfy

$$q_* p^* \left(\frac{dv}{y} \right) = -\frac{D_0 x(1+x)}{\alpha\xi} dx, \quad q_* p^* \left(\frac{v dv}{y} \right) = B_6(x) \frac{dx}{\xi}.$$

In both cases, the correspondence induces a surjective homomorphism

$$\Psi = p_* q^* : \text{Jac}(T_E) \longrightarrow \text{Jac}(C_E).$$

Here D_E is the disjoint union of the compact normalizations of all components of the correspondence, and the degrees and traces include all components.

We define the coefficients γ_j , $-3 \leq j \leq 3$, by

$$\begin{aligned} \gamma_{-3} &= \frac{3\sqrt{2}-4}{32}, & \gamma_{-2} &= -\frac{3(\sigma+1)(3\sqrt{2}-4)}{16\sigma}, \\ \gamma_{-1} &= \frac{24(\sqrt{2}-2)(\sigma^2+1) + (57\sqrt{2}-84)\sigma}{32\sigma^2}, & \gamma_0 &= \frac{2\sigma^3 + 3\sigma^2 + 3\sigma + 2}{2\sigma^3}, \\ \gamma_1 &= \frac{24(\sqrt{2}+2)(\sigma^2+1) + (57\sqrt{2}+84)\sigma}{32\sigma^3}, & \gamma_2 &= \frac{3(\sigma+1)(4+3\sqrt{2})}{16\sigma^3}, \\ \gamma_3 &= \frac{4+3\sqrt{2}}{32\sigma^3}. \end{aligned}$$

We define the coefficients c_j , $j \in \{-3, -1, 1, 3\}$, by

$$\begin{aligned} c_{-3} &= \frac{D_0^3(7-3\sqrt{5})}{8}, \\ c_{-1} &= \frac{D_0^3(-21+3\sqrt{5})}{8} - \frac{3BD_0}{2F} + \frac{3\sqrt{5}UD_0}{8F^2}, \\ c_1 &= \frac{D_0^3(21+3\sqrt{5})}{8} + \frac{3BD_0}{2F} + \frac{3\sqrt{5}UD_0}{8F^2}, \\ c_3 &= -\frac{D_0^3(7+3\sqrt{5})}{8}. \end{aligned}$$

The coefficient expansions of B_5 and B_6 are

$$B_5(r) = \frac{2}{3\alpha} \sum_{j=-3}^3 j\gamma_j r^{j+3}, \quad B_6(x) = \frac{-c_{-3} - c_{-1}x/3 + c_1x^2/3 + c_3x^3}{\alpha}.$$

The constant α in each polynomial is the constant of the corresponding pencil.

PROOF. We use the norm identity

$$P_r(z)P_r(-z) = Q_0(-z^2) - \mu(r)z^2.$$

First, we consider the quintic pencil. We define D_E by $P_r(z) = 0$ over T_E , and put

$$q(r, \xi, z) = (r, \xi), \quad p(r, \xi, z) = \left(-z^2, \frac{\alpha z \xi}{r^4}\right).$$

Since $P_r(0) = 1$, the coordinate z is nonzero. On D_E , the norm identity gives

$$Q_E(-z^2) = z^2(\mu(r) - E) = \frac{\alpha^2 z^2 \xi^2}{r^8},$$

so p takes values in C_E . The rational maps extend to morphisms on the compact normalizations. Both projections are nonconstant on each component. In fact, a constant value of z would give $Q_0(-z^2) - \mu(r)z^2 = 0$ for all r , which contradicts the nonconstancy of μ and $Q_0(0) = 1$. The projection q has total degree 5. For a fixed generic nonzero z , the polynomial $r^4 P_r(z)$ has degree 8 in r , with nonzero constant and leading coefficients. For a generic point (v, y) of C_E , each of the two choices $z = \pm\sqrt{-v}$ gives eight points, so p has total degree 16.

We denote the roots of P_r by $z_1, \dots, z_5$ and put $s_k(r) = \sum_j z_j^k$. Vieta's formulas give

$$s_1 = -\tau, \quad s_3 = -\tau^3 + \frac{3(\tau-1)(\sigma\tau-1)\zeta}{2\sqrt{2}\sigma^2} = \sum_{j=-3}^3 \gamma_j r^j.$$

Writing $t = \alpha\xi/r^4$, we have $v = -z^2$ and $y = zt$. Hence

$$\frac{dv}{y} = -\frac{2dz}{t}, \quad \frac{v dv}{y} = \frac{2z^2 dz}{t}.$$

Summing over the roots gives

$$q_* p^* \left(\frac{dv}{y} \right) = -\frac{2r^4}{\alpha\xi} ds_1, \quad q_* p^* \left(\frac{v dv}{y} \right) = \frac{2r^4}{3\alpha\xi} ds_3.$$

The constant coefficient of B_5 is $(4 - 3\sqrt{2})/(16\alpha) \neq 0$. The first trace has zero constant coefficient. The polynomial numerators of the two traces have degrees 4 and 6, respectively. Thus the two traces are linearly independent regular differentials on T_E .

Next, we consider the sextic pencil. Since $P_{-r}(z) = P_r(-z)$, we put $x = r^2$, $h = z/r$, and $\Pi_x(h) = P_r(rh)$. The polynomial Π_x has coefficients in $\mathbb{C}(x)$. We define D_E by $\Pi_x(h) = 0$ over T_E , and put

$$q(x, \xi, h) = (x, \xi), \quad p(x, \xi, h) = \left(-xh^2, \frac{\alpha h \xi}{x^2}\right).$$

The norm identity gives $Q_E(-xh^2) = \alpha^2 h^2 \xi^2 / x^4$, so p takes values in C_E . The projection q has total degree 6. The polynomial $r^5 P_r(z)$ has degree 10 in r for generic nonzero z . The simultaneous sign change $(r, z) \mapsto (-r, -z)$ identifies the two choices of $z = \pm\sqrt{-v}$, so p has total degree 10.

We use the same notation $s_k(r)$ for the sums of the k -th powers of the six roots of P_r . Newton's identities give

$$s_3(r) = \frac{7e(r)^3 - 12BF e(r) - 3(e(r)^2 - U)w(r)}{8F^3} = \sum_{j \in \{-3, -1, 1, 3\}} c_j r^j.$$

Consequently, we have

$$\sum_j h_j = D_0(1 - x^{-1}), \quad \sum_j h_j^3 = c_{-3}x^{-3} + c_{-1}x^{-2} + c_1x^{-1} + c_3.$$

Using the displayed formula for p , we obtain

$$\begin{aligned} \frac{dv}{y} &= -\frac{x^2}{\alpha\xi}(hdx + 2xdh), \\ \frac{v dv}{y} &= \frac{x^3}{\alpha\xi}(h^3dx + 2xh^2dh). \end{aligned}$$

Summing over the roots gives the stated sextic traces. The constant coefficient of B_6 is $-c_{-3}/\alpha \neq 0$. The first trace has zero constant coefficient. The polynomial numerators of the two traces have degrees 2 and 3, respectively. Thus the two traces are linearly independent regular differentials on T_E .

Finally, we consider the induced homomorphism of Jacobians. By pullback under q and pushforward under p , we obtain the homomorphism $\Psi = p_*q^*$ on degree-zero divisor classes; see [Mil86, Corollary 6.3]. We identify the cotangent spaces of the Jacobians with the spaces of regular differentials; see [Mil86, Propositions 2.1–2.2]. We compute the cotangent map of Ψ on an open set where the points of each fiber of q are distinct. Let A, A_0 be points of T_E . We represent the class $\Psi([A - A_0])$ by the sum of the points $p(P)$ with $P \in q^{-1}(A)$, minus the corresponding sum with $P \in q^{-1}(A_0)$. By differentiating the Abel integrals of a regular differential ω on C_E , we obtain the sum of $p^*\omega$ over these points. Therefore, the cotangent map is $\Psi^* = q_*p^*$. This equality extends across the branch points, since both sides are regular differentials. The differentials dv/y and $v dv/y$ form a basis of the regular differentials on C_E , and their traces are linearly independent. Hence Ψ^* is injective, so the image of Ψ has dimension 2 and equals $\text{Jac}(C_E)$. $\square$

6.6. Composition and induced polarizations

We compute the composition of the homomorphism in Theorem 6.11 with its adjoint. We put

$$J_C = \text{Jac}(C_E), \quad J_T = \text{Jac}(T_E), \quad f = \Psi^\dagger : J_C \longrightarrow J_T.$$

We denote the canonical principal polarizations by λ_C, λ_T , and their classes by $[\Theta_C], [\Theta_T]$. Pullback and norm are adjoint for these polarizations; see [KM16, §2, (2.1)–(2.3), and §4, (4.6)]. Thus $f = q_*p^*$. We use p_*, q_* for traces when they act on differentials.

We compute sums of odd powers of roots over the points above a fixed source point, using the possible choices of signs. For generic μ , each polynomial P satisfying the norm identity chooses one root from each pair of opposite roots. For generic nonzero z , we write $r_i(z)$ for the values of the parameter satisfying $P_{r_i(z)}(z) = 0$, counted with multiplicity. For each positive odd integer ℓ , we write $s_\ell(r)$ for the sum of the ℓ -th powers of the roots of P_r , counted with multiplicity. The fixed leading coefficient fixes the product of the signs. In the quintic case, the map

$r \mapsto \mu(r)$ has degree 16, and the formula $r = \zeta - \sigma\tau - \sigma - 1$ recovers r from the coefficients. Hence the curve defined by the norm equation realizes all sixteen sign choices. Fixing one chosen root z leaves eight choices, and each other root occurs four times with each sign. Thus $\sum_i s_\ell(r_i(z)) = 8z^\ell$ for every positive odd ℓ .

For the sextic pencil, we put $s^2 = \mu \neq 0$, $A(z) = H(-z^2) + sz$, and denote its roots by $w_1, \dots, w_6$. Newton's identities give $\sum_i w_i = \sum_i w_i^3 = 0$. The polynomial A has no zero or opposite roots, since $A(0) = 1$ and $A(z) - A(-z) = 2sz$. The selected factor P has leading coefficient F , opposite to the leading coefficient of A , so it is obtained by changing the signs of an odd number of roots. We put $\mathcal{L} = 4BF e + 4F^2 c - e^3$. The coefficient identities give

$$s_3 - \frac{1}{4}s_1^3 = -\frac{3\mathcal{L}}{4F^3}.$$

Changing the signs of one or five roots gives $\mathcal{L} = 0$. For three roots w_i, w_j, w_k , the left side is $6(w_i + w_j)(w_j + w_k)(w_k + w_i) \neq 0$. The factorization of the coefficient equation in (5.8) then gives $G = 0$. Thus the twenty choices obtained by reversing exactly three signs belong to the selected $G = 0$ component. The formulas

$$w = \frac{8F^2 c + e(e^2 + 4BF)}{e^2 - U}, \quad r = \frac{w - \sqrt{5}e}{q}$$

recover r generically, and $\deg \mu(r) = 20$, so all twenty choices occur. We obtain ten choices by requiring $w_1 = z$ to be a root of P . Each other root occurs four times with the positive sign and six times with the negative sign, so

$$\sum_i s_\ell(r_i(z)) = 12z^\ell - 2 \sum_{j=1}^6 w_j^\ell = 12z^\ell \quad (\ell = 1, 3).$$

The quotient by the simultaneous sign change accounts for the factor two in the trace in the reverse direction. These sums over choices of signs give the integers 16 and 12 multiplying the identity in the two compositions.

Then we have the following theorem.

Theorem 6.12. *Suppose that $\sigma \geq 1$, and that Q_E, S_E are squarefree. For the correspondences of Theorem 6.11, we have*

$$(6.5) \quad \Psi f = [m]_{J_C}, \quad m = \begin{cases} 16 & \text{for the quintic pencil,} \\ 12 & \text{for the sextic pencil.} \end{cases}$$

Consequently,

$$f^*[\Theta_T] = m[\Theta_C], \quad \Pi = \frac{1}{m}f\Psi \in \text{End}^0(J_T), \quad \Pi^2 = \Pi^\dagger = \Pi.$$

The degrees and traces include all compact normalized components of D_E .

PROOF. We compute the cotangent map

$$(\Psi f)^* = p_* q^* q_* p^*$$

on the basis $\omega_0 = dv/y$, $\omega_1 = vdv/y$ of $H^0(C_E, \Omega_{C_E}^1)$. For generic nonzero z , we put

$$F_z(r) = \begin{cases} r^4 P_r(z) & \text{for the quintic pencil,} \\ r^5 P_r(z) & \text{for the sextic pencil} \end{cases} = \sum_{j=0}^{n_F} A_j(z) r^j,$$

where $n_F = 8$ or 10 , respectively. We use the same notation $r_i(z)$ for all roots of F_z , and put

$$R_j(z) = \sum_i r_i(z)^j, \quad u_j = \frac{A_j(z)}{A_0(z)}, \quad v_j = \frac{A_{n_F-j}(z)}{A_{n_F}(z)} \quad (1 \leq j \leq 3).$$

The constant and leading coefficients are nonzero multiples of z . Newton's identities for the roots and their reciprocals give

$$(6.6) \quad \begin{aligned} R_{-1} &= -u_1, & R_{-2} &= u_1^2 - 2u_2, & R_{-3} &= -u_1^3 + 3u_1u_2 - 3u_3, \\ R_1 &= -v_1, & R_2 &= v_1^2 - 2v_2, & R_3 &= -v_1^3 + 3v_1v_2 - 3v_3, & R_0 &= n_F. \end{aligned}$$

Each of the functions u_j, v_j is a polynomial in z of degree at most j . We put

$$\mathcal{H}_\ell(z) = \sum_i s_\ell(r_i(z)), \quad \ell = 1, 3,$$

where $s_\ell(r)$ is the Newton sum of the roots of P_r . Substituting the coefficient formulas from (4.3) and (5.3) into (6.6), we obtain

$$(6.7) \quad \begin{array}{c|ccc} & n_F & \mathcal{H}_1(z) & \mathcal{H}_3(z) \\ \hline \text{quintic} & 8 & 8z & 8z^3 \\ \text{sextic} & 10 & 12z & 12z^3. \end{array}$$

For $\mathcal{H}_1$, we use $s_1 = -\tau$ and $s_1 = -e/F$, respectively. For $\mathcal{H}_3$, we use

$$\mathcal{H}_3 = \begin{cases} \sum_{j=-3}^3 \gamma_j R_j & \text{for the quintic pencil,} \\ c_{-3}R_{-3} + c_{-1}R_{-1} + c_1R_1 + c_3R_3 & \text{for the sextic pencil.} \end{cases}$$

The coefficients of $1, z, z^2, z^3$ are $0, 0, 0, 8$ and $0, 0, 0, 12$, respectively. We now apply Proposition 6.5 to the two differentials. On the curve defined by $P(z) = 0$, (6.4) gives

$$q^*(q_*p^*\omega_k) = \frac{2(-1)^{k+1}z}{(2k+1)y} ds_{2k+1}, \quad k = 0, 1,$$

where we use the cover with $x = r^2, z = rh$ in the sextic case. For each of the choices $z = \pm\sqrt{-v}$, we sum over all $r_i(z)$ and use $dv = -2zdz$. In the quintic case, the two choices give sixteen points of D_E . In the sextic case, the map $(r, z) \mapsto (-r, -z)$ identifies the twenty points on the cover in pairs, so we divide the sum by two. Putting $\delta = 1$ or 2 , respectively, we obtain

$$p_*q^*q_*p^*\omega_k = \frac{(-1)^k}{(2k+1)\delta} (\mathcal{H}'_{2k+1}(z) + \mathcal{H}'_{2k+1}(-z))\omega_0.$$

Here the primes indicate that we differentiate with respect to the argument before evaluating at z or $-z$. By (6.7), the matrices of the cotangent map in the two cases are $16I_2$ and $12I_2$. Both calculations hold on the open set where the projections are unramified. The traces are regular differentials on the compact normalizations, so the equalities extend to the complete curves. The sum includes every component of D_E . Since a homomorphism of complex abelian varieties is determined by its tangent map, these matrices prove (6.5).

The adjoint relation gives $f^\vee \lambda_T = \lambda_C \Psi$. Hence

$$\lambda_{f^*\Theta_T} = f^\vee \lambda_T f = \lambda_C[m],$$

which proves the equality of polarization classes. Finally,

$$\Pi^2 = \frac{1}{m^2} f(\Psi f) \Psi = \Pi, \quad \Pi^\dagger = \frac{1}{m} f \Psi = \Pi.$$

□

We now describe the finite kernel and the complementary abelian subvariety. We put

$$K = \ker f, \quad h_f = \#K, \quad U = \operatorname{im} f, \quad V = (\ker \Psi)^0,$$

and write $f = i f'$, where $i : U \hookrightarrow J_T$ is the inclusion and $f' : J_C \rightarrow U$ is the induced map. We denote the induced polarization by $\lambda_U = i^\vee \lambda_T i$. We define the two addition maps by

$$a : U \times V \longrightarrow J_T, \quad (u, v) \longmapsto u + v, \quad \tilde{a} : J_C \times V \longrightarrow J_T, \quad (c, v) \longmapsto f(c) + v.$$

Then we have the following corollary.

Corollary 6.13. *Under the assumptions of Theorem 6.12, the group K is a finite isotropic subgroup of $J_C[m]$ for the m -Weil pairing, and $h_f \mid m^2$. Moreover,*

$$(6.8) \quad \operatorname{im}(m \operatorname{Id}_{J_T} - f \Psi) = V, \quad \pi_0(\ker \Psi) \simeq \ker((f')^\vee), \quad \#\pi_0(\ker \Psi) = h_f.$$

We have

$$U \cap V = \ker \lambda_U, \quad \deg \lambda_U = \frac{m^4}{h_f^2}.$$

The maps $a, \tilde{a}$ are isogenies, with

$$(6.9) \quad \begin{aligned} \ker a &= \{(u, -u) : u \in U \cap V\}, & \deg a &= \frac{m^4}{h_f^2}, \\ \ker \tilde{a} &= \{(c, -f(c)) : c \in J_C[m], f(c) \in V\}, & \deg \tilde{a} &= \frac{m^4}{h_f}. \end{aligned}$$

The dimension of V is 5 in the quintic case and 2 in the sextic case.

PROOF. The identity $\Psi f = [m]$ makes df injective and gives $K \subset J_C[m]$. We write $J_C = W_C/\Lambda_C$, $J_T = W_T/\Lambda_T$, and denote by $\tilde{f}$ the linear map between the covering vector spaces that induces f . We denote the Riemann forms of λ_C, λ_T by $\mathcal{E}_C, \mathcal{E}_T$, respectively. The equality $f^*[\Theta_T] = m[\Theta_C]$ gives

$$\mathcal{E}_T(\tilde{f}u, \tilde{f}v) = m\mathcal{E}_C(u, v).$$

If u, v represent points of K , their images lie in Λ_T . Thus $m\mathcal{E}_C(u, v)$ is an integer, and the m -Weil pairing e_m satisfies

$$e_m(u \bmod \Lambda_C, v \bmod \Lambda_C) = \exp(2\pi i m \mathcal{E}_C(u, v)) = 1.$$

The pairing on $J_C[m]$ is perfect. Its restriction to each primary component shows that an isotropic subgroup has order dividing m^2 , since $\dim J_C = 2$.

The image of $m \operatorname{Id}_{J_T} - f \Psi$ lies in V . On V , this homomorphism equals $[m]_V$, so its image is all of V . We put $j = i^\vee \lambda_T : J_T \rightarrow U^\vee$. The dual of the inclusion i has connected kernel $(J_T/U)^\vee$. Hence $\ker j$ is connected. Since

$$(f')^\vee j = f^\vee \lambda_T = \lambda_C \Psi,$$

we have $\ker j = V$ and

$$\ker \Psi/V \simeq \ker((f')^\vee).$$

The isogenies f' and $(f')^\vee$ have degree h_f . This proves (6.8). The restriction of j to U is λ_U , so $U \cap V = \ker \lambda_U$. Also,

$$(f')^\vee \lambda_U f' = \lambda_C[m].$$

Taking degrees gives $h_f^2 \deg \lambda_U = m^4$.

The tangent spaces of U and V form a direct sum, since $\Psi f = [m]$. Thus a is an isogeny. Its kernel and degree follow from $U \cap V = \ker \lambda_U$. The factorization

$$\tilde{a} = a \circ (f' \times \text{Id}_V)$$

gives the degree of $\tilde{a}$. The kernel of $\tilde{a}$ has the form in (6.9), since $f(c) \in V$ implies $[m]c = \Psi f(c) = 0$. $\square$

6.7. Division of the homomorphism by two

We compute the action of $f = \Psi^\dagger$ on $J_C[2]$ and divide f by two in $\text{Hom}(J_C, J_T)$. We retain the correspondences and notation of Theorems 6.11 and 6.12. We denote the ramification divisor of the hyperelliptic projection $T_E \rightarrow \mathbb{P}^1$ by $\mathcal{R}_T$. This divisor is the sum of 16 finite points in the quintic case and 10 finite points in the sextic case.

Then we have the following proposition.

Proposition 6.14. *Suppose that $\sigma \geq 1$, and that Q_E, S_E are squarefree. For each finite Weierstrass point W of C_E , we have the equality of effective divisors*

$$(6.10) \quad q_* p^* W = \mathcal{R}_T.$$

Moreover, $J_C[2] \subset \ker f$, and there is a unique homomorphism $f_0 : J_C \rightarrow J_T$ such that

$$\begin{aligned} f &= 2f_0, & \Psi &= 2\Psi_0, & \Psi_0 &= f_0^\dagger, \\ \Psi_0 f_0 &= [n], & n &= \begin{cases} 4 & \text{for the quintic pencil,} \\ 3 & \text{for the sextic pencil.} \end{cases} \end{aligned}$$

The homomorphism f_0 satisfies

$$f_0^*[\Theta_T] = n[\Theta_C], \quad \Pi = \frac{1}{n} f_0 \Psi_0.$$

PROOF. First, we consider the quintic pencil. We write $W_i = (v_i, 0)$ for the five finite Weierstrass points, and $B_a = (r_a, 0)$ for the sixteen ramification points of T_E . All v_i, r_a are nonzero, since $Q_E(0) = S_E(0) = 1$. At B_a , we have $\mu(r_a) = E$ and

$$P_{r_a}(z)P_{r_a}(-z) = Q_E(-z^2).$$

The polynomial on the right has ten distinct nonzero roots. Exactly one of $\sqrt{-v_i}, -\sqrt{-v_i}$ is a root of P_{r_a} , and this root is simple. Thus there is exactly one point of the correspondence above B_a that maps to W_i . The coordinate ξ is a local parameter on T_E at B_a , since $S'_E(r_a) \neq 0$. The simple root of P_{r_a} gives one local graph $z = z(\xi)$ on the normalization. On this graph,

$$y = \frac{\alpha z(\xi)}{r(\xi)^4} \xi$$

has a nonzero linear coefficient. Since $Q'_E(v_i) \neq 0$, the coordinate y is a local parameter on C_E at W_i . Hence the ramification index of p is one at each of these sixteen points. The degree of p , including all components, is sixteen, so these points

exhaust $p^{-1}(W_i)$. There are no additional contributions at $r = 0$ or $r = \infty$. This proves (6.10) in the quintic case.

Next, we consider the sextic pencil in the quotient coordinates $x = r^2$, $h = z/r$. At a ramification point $B_a = (x_a, 0)$, the norm identity gives

$$\Pi_{x_a}(h)\Pi_{x_a}(-h) = Q_E(-x_a h^2).$$

For each of the six finite Weierstrass points W_i , exactly one of the two roots $h = \pm\sqrt{-v_i/x_a}$ belongs to Π_{x_a} . The root is simple, and

$$y = \frac{\alpha h}{x^2} \xi$$

again gives a nonzero linear coefficient. Each of the ten points above W_i has multiplicity one, and these points exhaust the degree-ten fiber of p . The quotient coordinates already identify the simultaneous sign change, so we count ten points. This proves (6.10) in the sextic case. Both arguments apply to every smooth fiber and include every normalized component. In particular, $f([W_i - W_j]) = 0$ for all finite Weierstrass points.

For the quintic pencil, we denote the point at infinity by W_∞ , and put $a_i = [W_i - W_\infty]$. The principal divisors

$$\operatorname{div}(v - v_i) = 2W_i - 2W_\infty, \quad \operatorname{div}(y) = \sum_{i=1}^5 W_i - 5W_\infty$$

give $2a_i = 0$ and $\sum_i a_i = 0$. The images $f(a_i)$ have one common value A . Thus $2A = 5A = 0$, and $A = 0$. The classes a_i generate $J_C[2]$. To verify this, we consider a function with its only pole at W_∞ , of order at most two. Such a function is a linear combination of $1, v$. Such a function cannot have a simple zero at one Weierstrass point or simple zeros at two distinct Weierstrass points. Using $\sum_i a_i = 0$, we reduce every other possible relation to a subset of one or two points. Hence the only relation among the five classes a_i is their sum, and they generate sixteen distinct classes. Since $\#J_C[2] = 16$, these classes give all of $J_C[2]$. For the sextic pencil, sending one of the six finite Weierstrass points to infinity gives a quintic model. The same argument shows that differences of the six finite points generate $J_C[2]$. Therefore $J_C[2] \subset \ker f$ in both cases.

Finally, we descend f through the quotient $[2] : J_C \rightarrow J_C/J_C[2] \simeq J_C$. The quotient gives a unique algebraic homomorphism f_0 with $f = f_0 \circ [2] = 2f_0$. Taking adjoints gives $\Psi = 2f_0^\dagger$. The identity $\Psi f = [m]$ gives $4f_0^\dagger f_0 = [m]$, where $m = 16, 12$. A homomorphism from a connected abelian variety to its finite 4-torsion subgroup is zero. We therefore cancel four in the Hom group and obtain $f_0^\dagger f_0 = [n]$, where $n = m/4$. The adjoint relation gives $f_0^\vee \lambda_T f_0 = \lambda_C[n]$, which proves the polarization equality. Substitution into $\Pi = f\Psi/m$ gives the last equality. $\square$

We put $K_0 = \ker f_0$, $\kappa = \#K_0$, and define

$$\tilde{a}_0 : J_C \times V \longrightarrow J_T, \quad (c, v) \longmapsto f_0(c) + v,$$

where $U = \operatorname{im} f_0 = \operatorname{im} f$ and $V = (\ker \Psi)^0$. We retain the addition map $a : U \times V \rightarrow J_T$ and the induced polarization λ_U . Then we have the following lemma.

Lemma 6.15. *Under the assumptions of Proposition 6.14, the group K_0 is isotropic in $J_C[n]$, and*

$$\kappa \mid n^2, \quad \kappa < n^2.$$

In particular, $\kappa \leq 8$ for the quintic pencil, and $\kappa \in \{1, 3\}$ for the sextic pencil.

PROOF. The argument using Riemann forms in Corollary 6.13, applied to $f_0^*[\Theta_T] = n[\Theta_C]$, gives $K_0 \subset J_C[n]$, isotropy, and $\kappa \mid n^2$. It also gives $\deg \lambda_U = \deg a = n^4/\kappa^2$. If $\kappa = n^2$, the addition map a for the complementary abelian subvarieties would identify the principally polarized Jacobian J_T with a product of two positive-dimensional principally polarized abelian varieties. For a smooth connected curve of genus g , the canonical theta divisor is the image of the $(g-1)$ -st symmetric power of the curve and is irreducible; see [Mil86, Theorems 5.1(a) and 6.6]. Every effective representative of its principal polarization is a translate of this divisor. The theta divisor of a nontrivial polarized product is reducible, which is a contradiction. Thus $\kappa < n^2$. $\square$

6.8. Determination of the kernel by monodromy

We determine the kernel $\ker f_0$ by monodromy of the source curves. We fix $\sigma \geq 1$ and denote by B_σ the set of $E \in \mathbb{C}$ for which Q_E, S_E are squarefree. This set is the complement of finitely many points. We retain $K_0 = \ker f_0$, $n = 4, 3$, and the notation of Proposition 6.14.

Then we have the following lemma.

Lemma 6.16. *Over B_σ , the homomorphisms f_0 define a morphism of integral homology local systems. The groups $K_0 \subset J_C[n]$ form a monodromy invariant finite sublocal system. Removing the additional singular values of T_E does not reduce the monodromy image of the source family.*

PROOF. On a simply connected open subset of B_σ , we choose flat integral homology bases for the two smooth projective families. We use the basis $dv/y, v dv/y$ of relative holomorphic differentials on C_E . The trace formulas of Theorem 6.11 have coefficients independent of E , and their periods over flat cycles vary continuously. We write $P_C(E)$ for the 2×4 period matrix of the source curve and $L(E)$ for the $2 \times 2g(T_E)$ matrix of periods of the two traces on the target curve. The real matrix

$$R_C(E) = \begin{pmatrix} \operatorname{Re} P_C(E) \\ \operatorname{Im} P_C(E) \end{pmatrix}$$

is invertible. In fact, a real homology class whose holomorphic periods vanish also has vanishing antiholomorphic periods, so it is zero by the Hodge decomposition and the nondegenerate period pairing. We therefore obtain the following matrix for the action of the already defined homomorphism Ψ_E on homology:

$$M_\Psi(E) = R_C(E)^{-1} \begin{pmatrix} \operatorname{Re} L(E) \\ \operatorname{Im} L(E) \end{pmatrix}.$$

This matrix is continuous and integral, hence locally constant. The intersection forms of the canonical polarizations are constant in flat bases, so the matrix M_f representing the adjoint homomorphism is also locally constant. Proposition 6.14 gives a unique integral matrix $M_{f_0} = M_f/2$ in every fiber. These matrices agree on overlaps and define the asserted morphism of local systems. On n -torsion, K_0 is the kernel of M_{f_0} modulo n , so it is monodromy invariant. This argument uses the explicit traces and does not require simultaneous normalization of the components of the correspondence.

The locus where the source curves are smooth contains B_σ , with only finitely many additional points. Loops generating the fundamental group of this locus

can be chosen to avoid these points. Thus the inclusion induces a surjection on fundamental groups, and the image of the monodromy representation for the source family is unchanged. $\square$

Then we have the following theorem.

Theorem 6.17. *For either specified pencil, suppose that $\sigma \geq 1$, $E \in \mathbb{C}$, and Q_E, S_E are squarefree. Then*

$$\ker f_0 = \{0\}.$$

PROOF. First, we consider the quintic pencil. Using B, e, Δ of (4.1), we put

$$C_\sigma = \frac{4\sigma(2\sigma^2 + \sigma + 1)}{(\sigma + 1)^3}, \quad A_\sigma(v) = 1 + Bv + ev^2, \quad T_\sigma(v) = 1 + C_\sigma v + (e/\sigma)v^2.$$

Coefficient comparison gives

$$(6.11) \quad vQ'_0(v) - Q_0(v) = A_\sigma(v)T_\sigma(v)(4\sigma v - 1).$$

For the degree-five map $\varphi_\sigma(v) = -Q_0(v)/v$, the roots of A_σ, T_σ have critical values $0, \Delta$, respectively. The latter assertion also follows from $Q_0(v) + \Delta v = (1 + \sigma^2 v)T_\sigma(v)^2$. The remaining critical point and its value are

$$v_* = \frac{1}{4\sigma}, \quad E_* = -\frac{(4\sigma + 1)(4\sigma^3 + 9\sigma^2 + 10\sigma + 2)^2}{4(\sigma + 1)^6} < 0.$$

Since $A_\sigma(v_*), T_\sigma(v_*) > 0$, the critical point v_* is simple. Its value differs from $0, \Delta$, which are nonnegative. Thus its local monodromy is a transposition, including when $\sigma = 1$. The connected degree-five covering has transitive monodromy. A transitive group of prime degree is primitive. The conjugates of a transposition give a graph whose connected components are blocks; primitivity makes this graph connected, and its edge transpositions generate S_5 . Consequently, the permutations induced by continuation of the roots around loops form S_5 . By Lemma 6.16, continuation around loops in B_σ induces the same permutations of the roots.

The point at infinity is a fixed Weierstrass section of the quintic family. The usual description by differences of Weierstrass points identifies $J_C[2]$ with the even-weight subspace of $\mathbb{F}_2^5$. A nonzero vector has weight two or four. The orbit of a weight-two vector spans this subspace, and differences of weight-four orbit vectors give weight-two vectors. Thus this four-dimensional S_5 -module is irreducible. If $K_0 \neq 0$, then $K_0 \subset J_C[4]$ has nonzero two-torsion. Monodromy invariance forces $K_0[2] = J_C[2]$, so $\#K_0 \geq 16$, contrary to Lemma 6.15.

We also record the discriminant obtained from (6.11):

$$\text{disc}_v Q_E = 256e^6 E^2 (E - \Delta)^2 (E - E_*).$$

The resultants of $A_\sigma, T_\sigma, 4\sigma v - 1$ with Q_E are $e^4 E^2$, $(e/\sigma)^4 (E - \Delta)^2$, and $(4\sigma)^4 (E - E_*)$. Their product divided by the leading coefficient e^2 of Q_E is its discriminant.

Next, we consider the sextic pencil, with $H(v) = 1 + Av + Bv^2 + Fv^3$ and $F > 0$. We make the base change and coordinate change

$$E = t^{-5}, \quad u = tv, \quad w = t^3 y.$$

The equation becomes

$$w^2 = (Fu^3 + Btu^2 + At^2 u + t^3)^2 + u.$$

At $t = 0$, the polynomial is $u(F^2u^5 + 1)$. Its derivative is 1 at zero and -5 at each of its other five roots. In the coordinates $s = 1/u, W = w/u^3$, we have

$$W^2 = (F + Bts + At^2s^2 + t^3s^3)^2 + s^5.$$

At $s = t = 0$, the two points $W = \pm F$ are smooth. With these two charts, the double cover of $\mathbb{P}^1$ gives a smooth proper genus-two family near $t = 0$. We put $\zeta = \exp(2\pi\sqrt{-1}/5)$. The deck action is

$$(t, u, w) \mapsto (\zeta t, \zeta u, \zeta^3 w).$$

On the holomorphic differentials $du/w, u du/w$, its eigenvalues on the fiber at zero are ζ^3, ζ^4 . Together with their complex conjugates, these give the characteristic polynomial on first cohomology

$$\Phi_5(X) = X^4 + X^3 + X^2 + X + 1.$$

Transport in this smooth proper family identifies this action, or its inverse according to the loop orientation, with the monodromy of the source family at infinity. Since the multiplicative order of 3 modulo 5 is four, Φ_5 is irreducible over $\mathbb{F}_3$. The action on $J_C[3]$, equivalently on first homology modulo three, is therefore irreducible. Lemma 6.15 gives $\#K_0 = 1$ or 3. A kernel of order three would be a nonzero proper invariant subspace of $J_C[3]$, contrary to Lemma 6.16 and irreducibility. Thus $K_0 = 0$ in both families. $\square$

We write λ_V for the restricted polarization on V . Then we have the following corollary.

Corollary 6.18. *Under the assumptions of Theorem 6.17, the map $f_0 : J_C \rightarrow J_T$ is an embedding, and*

$$\begin{aligned} \ker f &= J_C[2], & \ker \Psi_0 &= V, & \#\pi_0(\ker \Psi) &= 16, \\ \text{type}(\lambda_U) &= (n, n), & \text{type}(\lambda_V) &= \begin{cases} (1, 1, 1, 4, 4) & \text{quintic,} \\ (3, 3) & \text{sextic,} \end{cases} \\ U \cap V &= f_0(J_C[n]) \simeq (\mathbb{Z}/n\mathbb{Z})^4. \end{aligned}$$

The maps $a : U \times V \rightarrow J_T$ and $\tilde{a}_0 : J_C \times V \rightarrow J_T$ have degree n^4 , namely 256 in the quintic case and 81 in the sextic case. The map $\tilde{a} : J_C \times V \rightarrow J_T, (c, v) \mapsto f(c) + v$, has degree $16n^4$, namely 4096 in the quintic case and 1296 in the sextic case, and

$$\ker a = \{(f_0(c), -f_0(c)) : c \in J_C[n]\},$$

$$\ker \tilde{a} = \{(c, -2f_0(c)) : c \in J_C[2n]\}.$$

There are no homomorphism g and integer $d \geq 2$ such that $f_0 = dg$, and the only positive integers dividing f in the group of homomorphisms are 1, 2.

PROOF. An injective homomorphism of complex abelian varieties is an isomorphism onto its image. Hence the pullback equality $f_0^*[\Theta_T] = n[\Theta_C]$ gives type (n, n) on U . The dual of an embedding has connected kernel, so Ψ_0 is surjective and its kernel equals V . Since $f = 2f_0$, its kernel is $J_C[2]$, and duality gives the stated number of connected components of $\ker \Psi$. For $u = f_0(c)$, we have $\Psi_0(u) = nc$. Thus $U \cap V = f_0(J_C[n])$, which gives the kernel of the first addition map and its order n^4 . The map $f_0 \times 1$ identifies $\tilde{a}_0$ with a . For the homomorphism $f = 2f_0$, $\Psi_0(2f_0(c)) = 2nc$, which gives the second kernel and its order $(2n)^4$. For abelian subvarieties that are complementary with respect to the principal polarization,

the kernels of the induced polarizations both identify with $U \cap V$, as in the lattice argument of Corollary 6.13. The nontrivial elementary divisors of the two restricted polarizations therefore coincide. Filling the remaining entries with ones gives the asserted type of λ_V . If $f_0 = dg$ with $d \geq 2$, then $J_C[d] \subset \ker f_0$, which is impossible. If $f = dg$ with even d , cancellation of two gives $d = 2$. For odd $d > 1$, a Bezout identity for $2, d$ would make f_0 divisible by d , which is also impossible. $\square$

6.9. A double cover and a map of degree five

We identify the choices of root signs for the quintic polynomial with a component of the construction of Kanev and Lange. We use the quintic data $Q_E, e, \alpha, P_r, \mu(r)$ of Theorems 4.1 and 6.11. Then we have the following proposition.

Proposition 6.19. *Suppose that $\sigma \geq 1$, $E \in \mathbb{C}$, and Q_E, S_E are squarefree for the quintic pencil. The smooth projective models of*

$$\begin{aligned} \tilde{C}_E : y^2 &= Q_E(-z^2), & C'_E : t^2 &= -Q_E(v)/v, \\ \tilde{C}_E &\xrightarrow{\pi} C'_E \xrightarrow{g} \mathbb{P}_t^1, & (z, y) &\mapsto (v = -z^2, t = y/z) \mapsto t \end{aligned}$$

have genera four and two, and $\deg \pi = 2$, $\deg g = 5$. Here t denotes the coordinate of this quotient curve. The map π is branched at the two points of C'_E above $v = \infty$, both of which lie over $t = \infty$. Over a generic t , we divide the choices of one lift z_j for each of the five points in $g^{-1}(t)$ into two sets of sixteen, according to $\prod_j z_j = \pm 1/e$. The normalization of the component with product $-1/e$ is T_E , with the map to $\mathbb{P}_t^1$ given by $t = \alpha\xi/r^4$.

PROOF. Since $Q_E(0) = 1$, substituting $-z^2$ preserves squarefreeness and gives a polynomial of degree ten. Thus $\tilde{C}_E$ has genus four. The quadratic extension defining C'_E is branched at the five roots of Q_E and at $v = 0$, so C'_E has genus two. There are two points above $v = \infty$, since the pole order of $-Q_E(v)/v$ there is four. The involution of π is $(z, y) \mapsto (-z, -y)$. It exchanges the two points at $z = 0$. In the chart $h = 1/z, Y = y/z^5$, it acts by $(h, Y) \mapsto (-h, Y)$, so the two points at infinity are fixed. There are no other fixed points. The pole divisor of t has multiplicity one at the point above $v = 0$ and multiplicity two at each point above $v = \infty$. Hence $\deg g = 1 + 2 + 2 = 5$.

For fixed t , the five coordinates v_j are the roots of Q_{E+t^2} . Each has two lifts with $z_j^2 = -v_j$, giving thirty-two choices. Since the constant and leading coefficients of Q_{E+t^2} are $1, e^2$, we have $(\prod_j z_j)^2 = 1/e^2$. The normalization $P_r(0) = 1$ and leading coefficient e give $\prod_j z_j = -1/e$. Using the expression for r in terms of the coefficients given before Theorem 6.12 and $\deg \mu = 16$, we realize all sixteen choices with this product. The equation $t^2 = \mu(r) - E$ becomes $t = \alpha\xi/r^4$ on T_E . Since S_E is squarefree, this curve is connected. This expression identifies the function field of the selected component with that of T_E , which gives the asserted normalization. $\square$

Remark 6.20. The construction by choosing one lift of each root in Proposition 6.19 is given in [KL08, §§2.13–2.14, author manuscript arXiv:0707.0364v1]. Remark 3.9 in that manuscript extends the constructions and correspondence identities of §§3.2–3.8 to arbitrary B_n -coverings, allowing the nonsingular curves in the construction to be reducible. For the maps in Proposition 6.19, the double cover is ramified and its two branch points have the same image under g . Thus the etale

and simple ramification assumptions of [KL08, Proposition 4.6, the same author manuscript] fail, so its polarization conclusion does not apply directly. For these pencils, we give an explicit rational component of the curve defined by the norm equation and its full real path, together with the kernel and polarization calculations of Theorem 6.17 and Corollary 6.18.

Remark 6.21. The division $f = 2f_0$ takes place in the group of homomorphisms of abelian varieties. It does not divide the effective correspondence coefficient by coefficient. The kernel determination concerns the specified homomorphism. We do not determine which multiplication maps arise from compositions of other homomorphisms or correspondences.

Remark 6.22. If the induced polarization on U has type (d_1, d_2) , then $d_1 d_2 = m^2/h_f$. Corollary 6.18 gives $K = J_C[2]$, $h_f = 16$, and type (n, n) on U . The original map f has a nontrivial kernel, but f_0 is an embedding. The integer m refers to the composition for the specified correspondence; we do not determine which multiplication maps arise from other correspondences. In particular, $U \cap \ker \Psi = f(J_C[m]) \simeq J_C[m]/K$ has order m^4/h_f , and can differ from $U \cap V$.

Remark 6.23. The squarefreeness assumption concerns the curve and Jacobian assertions. For example, the sextic fiber $E = \Delta$ is singular for $\sigma > 1$, even though it is positive on the positive real axis. The transformations of Theorems 4.1 and 5.1 do not require squarefree fibers. Corollary 6.13 gives the isogeny obtained by adding points of the two complementary abelian subvarieties. Corollary 6.18 determines the finite kernel of $\Psi^\dagger$ and the restricted polarization type of its image. We do not determine which multiplication maps arise from compositions of other homomorphisms or correspondences.

We use the following kernel for the second differential.

Lemma 6.24. *Suppose that $N = 5$ or 6 , $Q_0 \in \mathbb{R}[v]$, $\deg Q_0 = N$, $Q_0(v) > 0$ for $v \geq 0$, $Q_0(0) = 1$, and $Q_\mu(v) = Q_0(v) + \mu v$, where $\mu \geq 0$. We denote the roots of Q_μ by ρ_j and choose each square root $\sqrt{-\rho_j}$ with positive real part. We put $z_j = -\sqrt{-\rho_j}$ and $s_3(\mu) = \sum_j z_j^3$. Then*

$$\frac{ds_3}{d\mu} = \frac{3}{2\pi} \int_0^\infty \frac{v^{3/2}}{Q_\mu(v)} dv > 0.$$

PROOF. We first assume that the roots are simple. Differentiating $Q_\mu(\rho_j) = 0$ gives

$$\frac{ds_3}{d\mu} = -\frac{3}{2} \sum_j \frac{\rho_j \sqrt{-\rho_j}}{Q'_\mu(\rho_j)}.$$

Since $N \geq 3$, the partial fraction expansion of $v/Q_\mu(v)$ has residues whose sum is zero. Multiplying this expansion by v , we obtain

$$\frac{v^2}{Q_\mu(v)} = \sum_j \frac{\rho_j^2}{Q'_\mu(\rho_j)} \frac{1}{v - \rho_j}.$$

Apply the convergent beta integral [DLMF, (5.12.3)] with the factor $v^{-1/2}$. Since $\rho_j^2(-\rho_j)^{-1/2} = -\rho_j \sqrt{-\rho_j}$, we obtain the formula. For multiple roots, the contour integral defining $s_3(\mu)$ gives convergence of $ds_3/d\mu$ under the perturbation $Q_0(v) + \varepsilon v$. The integral is dominated by the unperturbed integrand, which has orders $v^{3/2}$

at zero and $v^{3/2-N}$ at infinity. Thus the formula extends to multiple roots by the argument of Lemma 3.1. $\square$

By this lemma, we obtain the following corollary.

Corollary 6.25. *Under the assumptions on the real paths in Theorems 4.1 and 5.1, we choose $y = \sqrt{Q_E(v)} > 0$ on $v \geq 0$. In the quintic case, we choose $\xi = \sqrt{S_E(r)} > 0$ for $0 < r < L$ and integrate from L to 0. In the sextic case, we choose $\xi = \sqrt{S_E(x)} > 0$ for $0 < x < L^2$ and integrate from L^2 to 0. For $k = 0, 1$, we have*

$$\int_0^\infty \frac{v^k dv}{\sqrt{Q_E(v)}} = \begin{cases} \int_L^0 q_* p^* \left(\frac{v^k dv}{y} \right), & \text{quintic case,} \\ \int_{L^2}^0 q_* p^* \left(\frac{v^k dv}{y} \right), & \text{sextic case.} \end{cases}$$

The traces on the right-hand side are the explicit differentials in Theorem 6.11. The integral identity also holds at nonsquarefree fibers in the real parameter ranges stated in the two theorems, where the right-hand side denotes the explicit differential formulas rather than a correspondence between smooth curves.

PROOF. We obtain the case $k = 0$ from Proposition 3.2 at $\beta = 1/2$. For $k = 1$, Lemma 6.24 and the beta integral give

$$\begin{aligned} \frac{2}{3} \int_0^\infty \lambda^{-1/2} \frac{ds_3}{d\mu} \Big|_{\mu=E+\lambda} d\lambda &= \frac{1}{\pi} \int_0^\infty v^{3/2} \int_0^\infty \frac{\lambda^{-1/2}}{Q_E(v) + \lambda v} d\lambda dv \\ &= \int_0^\infty \frac{v dv}{\sqrt{Q_E(v)}}. \end{aligned}$$

The positive integrands justify the interchange of integrals. Substitute $\lambda = \mu(r) - E$ and use the trace of $v dv/y$. For the sextic pencil, substitute $x = r^2$. The square root chosen for the integrals on the right is positive on the open interval, and $\mu'(L) \neq 0$ gives an integrable square-root singularity at the finite endpoint. $\square$

Remark 6.26. The real integrals in the corollary are taken along paths with endpoints; in particular, $v = 0$ is not a branch point of C_E . The corollary identifies real Abelian integrals rather than closed cycles. By the norm identity and the two independent traces, we obtain the correspondence and the Jacobian quotient.

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